HYDROL 15176
11 January 2006 Disk Used
ARTICLE IN PRESS
No. of Pages 8, Model 6+

Journal of Hydrology (2006) **xxx**, xxx–xxx

ELSEVIER

available at www.sciencedirect.com

SCIENCE DIRECT®

journal homepage: www.elsevier.com/locate/jhydrol

The texture of rain: Exploring stochastic micro-structure at small scales

A.B. Kostinski [a,*], M.L. Larsen [a], A.R. Jameson [b]

[a] *Department of Physics, Michigan Technological University, 1400 Townsend Drive, Houghton, MI 49931, USA*
[b] *RJH Scientific, Inc., El Cajon, CA 92020, USA*

Received 1 October 2002; received in revised form 26 November 2003; accepted 30 November 2005

Summary The main theme of this review is the importance of a discrete approach to describing rain at spatial scales comparable to inter-drop separation. We propose that the pair correlation function should be used to define and measure the texture of rain. To that end, we discuss the pivotal role of the Poisson process for examining this micro-structure of rain. The importance of statistical stationarity and the essential distinction between a Poisson distribution and a Poisson process are emphasized. It is argued that the correlation-fluctuation theorem (which relates drop count variance to the pair-correlation function) is ideally suited for scale-dependent exploration of rain micro-structure in the discrete ''shot noise'' limit. The likelihood of spurious negative correlations at fine spatial scales is pointed out as instruments are pushed to their resolution limits. One of the consequences is that possibly spurious Poisson statistics at a given spatial scale may result from a cancellation on sub-scales. We then proceed to examine implications of stochastic microstructure and show that the notion of spatially variable and random concentration (or size distribution) does not always provide an adequate description of rain texture.

Introduction

In this note, the words ''rain micro-structure'' are taken to the extreme of the spatial scales comparable to the mean separation between raindrops, i.e., down to a few centimeters. In this regime, the integral parameters such as liquid water content or rain rate are not always suitable and one must pay particular attention to fluctuations caused by the discrete nature of rain (similar to ''shot noise'' in physics, e.g., Van Kampen (1992)). It is the emphasis on ''the importance of being discrete'' which provides a unifying theme for this note. This is not the first time this issue has been raised, e.g., see a recent review of the ''large particle limit in rain'', (Lovejoy et al., 2003). There, the authors lament the paucity of studies aimed specifically at scale-dependence of rain microstructure. We quote (from Lovejoy et al., 2003):

> To date, very few small scale studies have attempted to systematically consider the statistics as functions of scale.

* Corresponding author. Tel.: +1 906 487 2580; fax: +1 906 487 2933.
E-mail addresses: alex_kostinski@mtu.edu (A.B. Kostinski), mllarsen@mtu.edu (M.L. Larsen).

0022-1694/$ - see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.jhydrol.2005.11.035

We agree. This note is an attempt to address this very question: scale-dependent texture at short distances. However, unlike Lovejoy et al. (2003), our goal here is to introduce a route to exploring scale-dependence of spatial correlations in rain which is free of *ad hoc* assumptions (unlike, for example, the fractal assumption, accompanied by the customary fitting of data to some power law and extracting a fractal dimension). Our mathematical tools are supplied by the theory of stochastic point processes and the approach is free of assumptions about the scaling of rainfall, except for statistical stationarity (homogeneity) – ''*the first and most common assumption [in hydrology and other geophysical sciences]*'' Bras and Rodriguez-Iturbe (1993, p. 4).

From the perspective of random point processes, the most popular stochastic model for the spatial and temporal distributions of raindrops and cloud droplets appears to be that of ''perfect randomness'' (''ideal gas''). In fact, physicists and mathematicians outside hydrology and meteorology often use rain as a standard of randomness against which to measure mysterious correlations of the quantum world. To take but one example, a recent quantum physics review article by Spence (2002, p. 377) begins thus

> Like the gentle patter of raindrops, we expect photons, the quanta of sun-light, to arrive at Earth at random intervals...

Likewise, countless probability texts consider raindrops striking the roof of a house a classic example of a Poisson process (e.g., Van Kampen, 1992, p. 34). Evidently, most physicists and mathematicians outside hydrology are under the impression that rain is devoid of microstructure. This is despite: (i) abundant evidence to the contrary provided by the vast literature on fractal rain characterization (e.g., see Bras and Rodriguez-Iturbe, 1993; Gupta and Waymire, 1990; Lovejoy and Schertzer, 1995; Marsan et al., 1996; Peters et al., 2002; Vaneziano et al., 1996; Waymire, 1985; Zawadzki, 1995); (ii) everyday observations of ''sheets'' of rain and other structural elements; (iii) observations and data analyses based on correlation theory of random processes (e.g., see Jameson and Kostinski, 1999, 2000). Let us then begin by defining the ''perfect randomness'' model more precisely so that deviations from it (structure) can be identified.

Poisson process and stochastic structure

Aside from the fractal method, there are two basic approaches to describing ''structure'' or ''patterns'' in random phenomena. One method relies on trends attributed to averages of otherwise random variables while the other employs the notion of a correlation function in order to describe a ''degree of order in a sea of randomness''. We adopt the latter approach and, rather than dwell on definitions, proceed to an example.

Consider the three panels of Fig. 1 containing point ''events'' (e.g., raindrops) frozen in time. The patterns, from left to right, are: perfect spatial randomness (homogeneous Poisson process); a clustered or spatially correlated pattern (homogeneous but not a Poisson process); and vertically stratified randomness (inhomogeneous Poisson process). Hence, perfect randomness requires the absence of ''trends'' as well as the absence of spatial correlations (e.g., see Shaw et al., 2002, for a tutorial summary in a meteorological context.) However, also note that the statistical homogeneity by itself need not preclude the existence of local clusters. For the rest of this paper, we will confine ourselves to statistically homogeneous (stationary) rain.[1]

It turns out that the clustered pattern (middle panel of Fig. 1) can often be understood as a statistically homogeneous field of fluctuating local concentration, sometimes referred to as a Cox or doubly stochastic process (Cox and Isham, 1980; Sasyo, 1965; Kostinski and Jameson, 1997, 2000). Note, however, that this interpretation requires a wide separation of the three scales: characteristic length of concentration variations, the scale on which spatial concentration is defined, and the mean inter-particle distance (e.g., see Friedlander, 2000, p. 7). On the other hand, random spatial patterns may also be more regular than perfect randomness (an obvious extreme case being a perfect lattice). The latter possibility can arise via ''anti-clustering'' caused by mutual particle repulsion such as reported, e.g., in Brenner (1999). This may also occur in mist, fog or drizzle where drops go around each other along nearly laminar streamlines – reflected in the raindrop coalescence efficiency being less than unity (see Chapter 15 of Pruppacher and Klett, 1997). Furthermore, raindrops possess definite size which naturally provides a length scale at which to expect exclusion of neighbors (negative spatial correlations) which will be defined precisely in the next section.[2]

The pair-correlation function

Let us consider spatial microstructure of rain as revealed by two-dimensional ''pavement patterns'' (rain flux imprints). It is not our goal to develop new theories of rain. Rather, the task is more modest: development of mathematical tools suitable for high spatial resolution rain analysis which can be performed in a scale-localizable manner but is free of ad hoc assumptions. For the sake of simplicity, we confine ourselves to monodisperse rain (drop size distributions are discussed later in the paper). Consider then the three patterns of Fig. 2 (1st row) which represent hypothetical traces left by raindrops arriving on a pavement during a brief time period.[3] We then ask: *Which spatial pattern is representative of real rainfall?* This question is not merely of academic interest but has implications in several fields, e.g., interception of raindrops by vegetation (Calder, 1996; Calder et al., 1996) or soil erosion.[4] The second row depicts rain flux imprints for a longer time period, representing 500 of the ''thin slices'' (falling raindrops of the

[1] A single realization of a homogeneous random process, whose duration is comparable to coherence time, might appear as an inhomogeneous one and it is, therefore, desirable to secure a much longer time series.

[2] Where we also show that disdrometers are likely to yield spurious negative correlations when pushed to their resolution limits.

[3] Early experimenters actually used dye paper and flour for drops size distribution measurements (Marshall and Palmer, 1948).

[4] See Chiu (1971); Clarke (1998) for an interesting tale about Einstein's view of rain pavement patterns – our inspiration for Fig. 2.

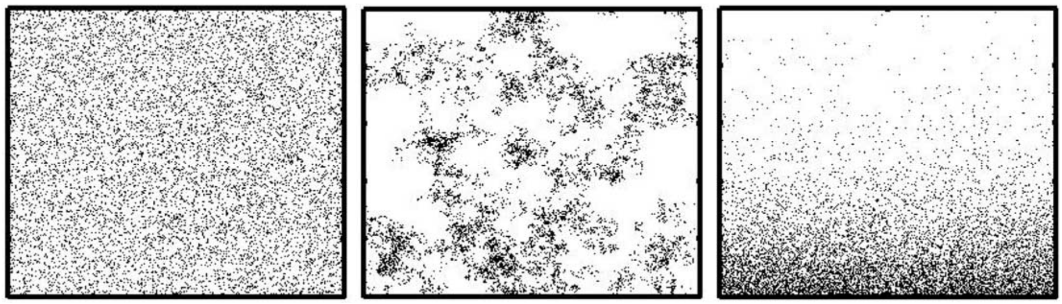

Figure 1 The left panel represents the ideal of randomness: statistically homogeneous Poisson process, characterized by a complete lack of spatial correlations. Statistically homogeneous but spatially correlated random process is depicted in the middle panel. Drop positions are uniformly distributed but are not independent random variables so that clump formation is allowed (but clump ''centers'' are uniformly distributed). The ''patchiness'' is quantified by the pair correlation function $\eta(l)$, specifically defined as a deviation from perfect randomness (Poisson process) as discussed in the text. For completeness, a statistically inhomogeneous Poisson process is illustrated in the right panel. There are no clumps but unlike the homogeneous process, number density here is a *deterministic* (exponentially decreasing) function of height.

same size). Why is there such a striking difference in the cumulative ''wetted'' area?

In order to answer the question quantitatively, we borrow from statistical physics a crucially important tool: the pair correlation function (pcf). To introduce the pcf, it is helpful to elaborate on the notion of perfect spatial randomness (middle column of Fig. 2). As stated earlier, complete lack of spatial correlations in drop positions is the defining feature of perfect randomness. This can be interpreted in several ways (e.g., see Cox and Isham, 1980; Iranpour and Chacon, 1988, p. 46 and Chapter 3, respectively): (i) given N points, droplet positions are uniformly, identically, and independently distributed random variables; (ii) nearest drop distances (areas in 2D, volumes in 3D, e.g., see Cox and Isham, 1980; Feller, 1966) are exponentially distributed; (iii) the probability of finding a given number of drops in a fixed volume is given by the Poisson distribution (for *any* volume). The difference between the three descriptions is in the choice of the random variable: raindrop position, inter-drop spacing, or number of drops in a volume.

Perhaps the most satisfying route to defining perfect randomness and deviations from it is based on the notion of the pcf (which is identically zero in the ideal case). The pcf is defined via

$$P(1,2) = c^2 \mathrm{d}V_1 \mathrm{d}V_2 [1 + \eta(l)], \qquad (1)$$

where $P(1,2)$ is the joint probability of finding a drop in each of the two disjoint volume elements $\mathrm{d}V_1$ and $\mathrm{d}V_2$, c is the number density, $c\mathrm{d}V \ll 1$ is the probability of finding a raindrop in $\mathrm{d}V$, $\eta(l)$ is the pair correlation function and l is the separation distance between the two elementary volumes (e.g., see Landau and Lifshitz, 1980).[5] For example, $\eta(l) = 3$ yields a factor of 4 enhancement of finding another drop, distance l away from a given drop. Likewise, $\eta(l) = -1$ represents impossibility of encountering another droplet distance l away from a given droplet (e.g., when l is less than a raindrop diameter). Thus, *perfect randomness is characterized by pcf identically equal to zero for all l.* The three basic types of pair correlation functions are depicted schematically in the 3rd row of Fig. 2.

We now return to the question of rain texture: *Is real rain most similar to left, middle, or right column of Fig. 2?* The answer is likely a mixture of the 3 ''states'', depending on the spatial scale but we can, at least, address the issue in a precise manner by measuring the pcf of rain flux imprints as indicated by Eq. (1). Note that $\eta = \eta(l)$ defines rain microstructure (texture) and does so in a scale localizable manner. No assumptions whatsoever are made here about η's functional form. It is completely general. For contrast, recall that (i) the often made (at least implicitly) assumption of perfect randomness requires that η vanish for all l; (ii) the fractal approach is *based on the assumption* of scale-invariance which, in our terminology, corresponds to a power-law functional form for $\eta(l)$ (Vicsek, 1989, p. 23; and Shaw et al., 2002).

Direct application of Eq. (1) is often problematic, however, as the joint probability function $P(1,2)$ (likelihood of drop pairs separated by distance l) must be estimated from data which, in turn, becomes increasingly sparse as the scale decreases. For that reason, we next introduce an integral measure which can be used as a smoother estimator of the pair-correlation function.

The correlation-fluctuation theorem 213

Given the increase in count fluctuations, typically caused by clustering (e.g., see 4th row of Fig. 2), it is natural to ask whether there is a relation between the strength of departure from perfect randomness (as measured by the pcf) and the deviation from the Poisson distribution. Indeed, such a connection exists. In the course of their studies of X-ray scattering by liquids, Ornstein and Zernike (1914) discovered that the mean squared fluctuation $(\delta N)^2$ of particle counts (variance of N) in a given volume is related to the pair correlation function integrated over the same volume. This ''fluctuation-correlation theorem'' is as follows (e.g., see Landau and Lifshitz, 1980, p. 352, Eq. 116.5).

[5] The volume elements here can be interpreted as, say, square mm of the pavement area and of a mm height so that no two drops can be in the same $\mathrm{d}V$ at any one time. For a link between the pair-correlation function to the more familiar autocorrelation function, see Shaw et al. (2002, pp. 1049–1050).

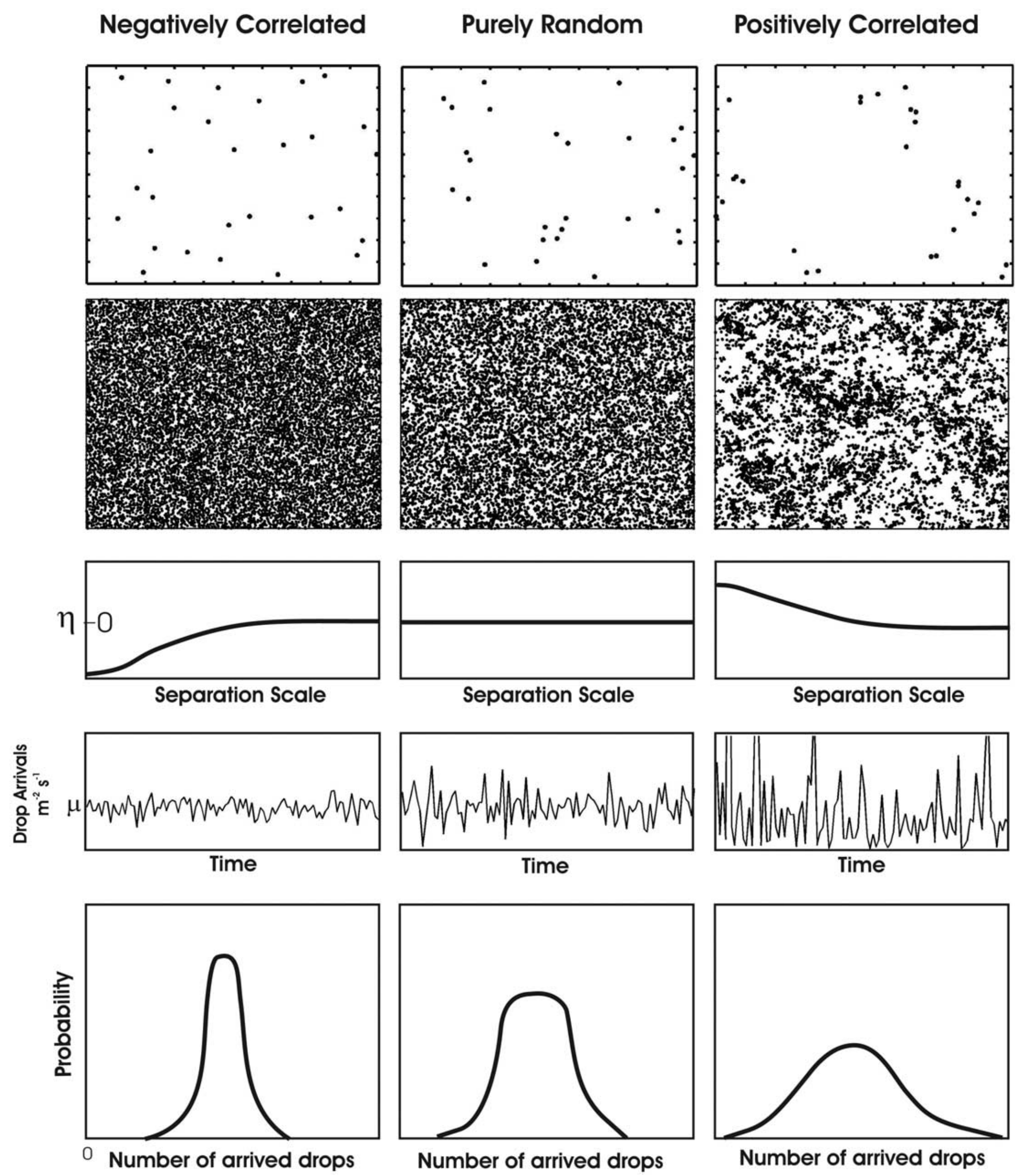

Figure 2 Imprints left by fallen raindrops after a brief time interval. Left column corresponds to ''negatively correlated raindrops, the middle column represents a purely random rain flux, and the right column corresponds to a positively correlated (clustered) rain. The next row shows the surface after raining for a longer period, representing 500 of the ''thin slices'' used in the first row (first row dots were expanded to enhance visibility). Identical size (hence, no differential velocity) and number of raindrop splashes were used in all cases but clearly the effective area coverage differs greatly. The third row shows schematic pair correlation functions characterizing the three distributions. The fourth row displays the corresponding time series of total raindrop counts arriving per unit area and time. Note fluctuations increasing from left to right. The bottom row displays the same raindrop counts as a histogram. (In accordance with the correlation-fluctuation theorem, the negatively correlated medium has the narrowest distribution — see text for details.)

$$\frac{\overline{(\delta N)^2}}{\bar{N}} - 1 = \frac{\bar{N}}{V}\int_V \eta \, \mathrm{d}V, \qquad (2)$$

where $\bar{N} \equiv cV$ with c being the drop concentration, η is the pair correlation function between particle counts in some volume elements $\mathrm{d}V_1$ and $\mathrm{d}V_2$ within V and $\delta N \equiv N - \bar{N}$ is the deviation from the mean count in a given volume V. Again we stress that this relation is completely general (aside from stationarity, which is a requirement for *any* correlation function) and involves no assumptions about the random process (as opposed to, for example, power-law scaling used in fractal analysis). In the one-dimensional case (to be discussed shortly), Eq. (2) becomes:

$$\frac{\overline{(\delta N)^2}}{\bar{N}} - 1 = \frac{\bar{N}}{L}\int_0^L \eta(\ell)\, \mathrm{d}\ell, \qquad (3)$$

where $\bar{N} = \bar{N}(L)$ and $N = N(L)$.

The volume-averaged approximation to the pair correlation function can be calculated as

$$\bar{\eta} \equiv \frac{1}{V}\int_V \eta \mathrm{d}V = \frac{\overline{(\delta N)^2}}{[\bar{N}(V)]^2} - \frac{1}{\bar{N}(V)}, \quad (4)$$

where we emphasize the explicit volume dependence of $\bar{N}$. The $\bar{\eta}$ is straightforward to calculate from measurements of $\bar{N}$ and $\overline{(\delta N)^2}$ versus volume and it provides a smoother (but coarser) alternative to the fundamental definition of η.

Note that in the limiting case of no correlation in (2), $\eta(l) \equiv 0$ and we recover the Poisson relation $\overline{(\delta N)^2} = \bar{N}$. The fourth and fifth row of Fig. 2 illustrate this. Both the time series and the histograms display the increase of variability (variance) as the integral of the pair correlation function over a counting volume increases from negative values (left), through zero (middle), to positive values (right) – all in accordance with the correlation-fluctuation theorem. As the correlation integral increases, so do the fluctuations – hence the name of the theorem. Since the assumption of Poisson statistics is so prevalent and fundamental to most rain studies as well as to radar meteorology, we think that it is crucial to point out the following:

> the validity of the Poisson variance relation $\sigma^2 = \bar{N}$ for a given measurement volume V implies only that the integral in Eq. (2) vanishes. It does not imply that the pcf is zero at all scales (as in the Poisson process).

For example, for a long thin cylindrical integration volume such as the rain volume seen by a disdrometer in stationary conditions (or volume of raindrops detected by an optical probe during an aircraft traverse, Shaw et al. (2002)), Eq. (3) yields a one-dimensional integral

$$\int_0^L \eta(l)\,\mathrm{d}l = 0. \quad (5)$$

It is important to realize that the integral can vanish as a result of cancellation of positive and negative $\eta(l)$ contributions at different scales. This is the critical difference between the Poisson process and the Poisson distribution. The former requires $\eta(l) \equiv 0$, while the latter demands only that $\int_0^L \eta(l)\,\mathrm{d}l = 0$ hold for the length scale of interest (L). Therefore, depending on the resolution some experiments will pick up the non-Poissonian variance and some will not. (Higher moments are not considered here.)[6]

The above observation is particularly relevant to rain measurements obtained with disdrometers. As with most instruments, disdrometers have length and time scales (resolutions) below which they are unable to detect two drops (the equivalent of a ''dead time'' in counters) and create an artificial ''exclusion volume'' where $\eta = -1$. This, in turn, yields spurious negative correlations in drop positions/arrival times.[7] Such spurious correlations may negate real clustering and result in Poisson statistics on longer scales.

There may also be real physical causes for negative spatial correlations, although they are hardly ever mentioned in the literature. As was mentioned above, one might expect ''exclusion'' to occur at very small scales in mist, fog or drizzle where drops go around each other along nearly laminar streamlines. Such ''mutual avoidance'' follows from observations of raindrop coalescence efficiency considerably below unity (see Chapter 15 of Pruppacher and Klett, 1997).

[6] For example, having to assume Poisson statistics for, say, a typical radar resolution volume (e.g., Uijlenhoet et al., submitted) may be approximately accurate and not nearly as restrictive as the Poisson process assumption.

[7] Raindrops posses definite size (d) so that inter-drop separation must be larger. Because of this exclusion $\eta = -1$ for, at least, $l \leqslant d$.

Texture implications for integral parameters and the notion of a drop size distribution

How does the presence of fine scale texture ($\eta(l)$) affect our description of rain in terms of spatially continuously varying integral parameters such as rain rate or liquid water content? Can one simply ignore their ultimately discrete nature? To answer this, consider the simplest integral parameter notion, namely, that of spatially varying drop concentration denoted as $c(\mathbf{x})$. The expressions ''concentration inhomogeneity'' or ''concentration fluctuations'' are often used in the literature (Pruppacher and Klett, 1997) to describe the fact that $c(\mathbf{x})$ is treated as a random function of position. However, the notion of concentration fluctuations described by $c(\mathbf{x})$ implies a wide separation of three scales: inter-particle distance, scale on which concentration is defined and the characteristic scale over which concentration is varied. At the discrete level, drop number fluctuations in such a picture correspond to the so-called Cox (or doubly stochastic Poisson) process. If the relevant length scales are indeed widely separated, this can be visualized as the right column of Fig. 2 (e.g., see Kostinski and Jameson (2000)). Let us make this argument more precise.

To specify ''concentration inhomogeneities'', consider a distribution of similar patches (containing raindrops), roughly of a size L and relative voids of about the same size. Then, drop counts will obey the Poisson distribution as long as the local concentration (c) remains constant. However, on longer spatial scales (larger than L), the concentration itself will fluctuate as measurements move from patch to patch. Thus, to obtain the total (over many Ls) drop count distribution, one must integrate over $p(\bar{N})$

$$P(N) = \int_0^\infty P(N|\bar{N})p(\bar{N})\,\mathrm{d}\bar{N} = \int_0^\infty \frac{\bar{N}^N \exp(-\bar{N})}{N!}p(\bar{N})\,\mathrm{d}\bar{N}, \quad (6)$$

where the vertical bar denotes conditional probability, V is an individual measurement volume (assumed much smaller than L^3), and $\bar{N} = cV$. Hence, the process is doubly stochastic (Cox) because the ''shot noise'' fluctuations ride on top of the longer scale patch-to-patch fluctuations. Now, these sources of randomness are due to independent causes and their variances, therefore, add:

$$\sigma_N^2 = \sigma_P^2 + \sigma_{\bar{N}}^2. \quad (7)$$

As expected, the variance is increased beyond that of a pure Poisson pdf by the variance of $\bar{N} = cV$ (''concentration inhomogeneities''), that is, the first term is the pure Poisson contribution i.e., $\sigma_P^2 = \mu \equiv \int_0^\infty \bar{N}p(\bar{N})\,\mathrm{d}N$, and $\mu = E(\bar{N})$ is the expectation value of the raindrop counts when averaged over realizations for the entire domain. For example, when the concentration distribution is an exponential one, $\sigma_N^2 = \mu + \mu^2$ results (Kostinski and Jameson (2000)).

It can be seen immediately that any negatively correlated media cannot be described as a superposition of locally Poisson processes and therefore falls outside the ''concentration inhomogeneity'' framework. This can be seen by noting that the fluctuation-correlation theorem allows sub-Poisson variance when $\bar{\eta}$ is negative but the framework of concentration fluctuations does not (as shown above). This is illustrated in the 4th row of Fig. 2. Thus, unlike negatively correlated rain, the ''concentration inhomogeneity'' description always (at any scale) yields a super-Poissonian variance. In other words, if rain is negatively correlated on some length scale, fundamentally it cannot be adequately described via spatially varying liquid water content, rain rate, etc. until much longer scales are reached. How long? The scale (call it X) must be long enough so that the memory of negatively correlated η is ''erased'' from the integral $\int_0^X \eta(l)\,dl$.

Next, let us ask whether ''fine texture'' requires similar reconsideration of a *size-distributed rain*. We shall still assume statistically stationary and homogeneous rain, i.e., one with an ''equilibrium'' size distribution. Let us regard the normalized part of the drop size distribution (DSD) as a probability density function (e.g., see Kostinski and Jameson, 1999). For example, for the simplest exponential size distribution, we write

$$N(D) = N\left[\frac{1}{\bar{D}}\exp(-D/\bar{D})\right], \tag{8}$$

where N is the total number of drops per volume V and the *pdf* is the expression in square brackets (call it $p(D)$) because $\int_0^\infty p(D)\,dD = 1$. Then the probability of finding a drop size within a range (D, D + dD) is given by $p(D)\,dD$. In a manner, analogous to the spatially varying concentration above, we can now inquire about the nature and time evolution of a general size distribution probability density $p = p(D, \mathbf{r}, t)$. Insofar as the drop size distribution is a generalization of the spatially varying random concentration, the latter is subject to the same difficulties with regards to rain texture as the former. Time evolution of a size distribution is of additional concern, however.

The traditional approach to evolution of size-distributed rain is based on the coagulation equation which is an integro-differential equation for a general space and time varying random function $p(D, \mathbf{r}, t)$. Much of cloud physics is dominated by the idea that such a stationary size distribution evolves naturally with time as a result of *equilibrium between break-up and coalescence*, e.g., Atlas and Ulbrich (2000), Young (1993). But is this ''equilibrium'' notion compatible with the state of perfect spatial randomness or does it imply some time-dependent form of $\eta(l)$? Despite the arguments in Srivastava (1971), Valdez and Young (1985) that there might not be enough time for an equilibrium to be established in a real atmosphere, the idea still appears prevalent when interpreting observations as well as computer model results, e.g., see Ulbrich and Atlas (2002). To that end we shall next briefly re-examine the ''discrete limit'' to point out the following:

(1) Spatial correlations at sufficiently short scales cannot be incorporated into the coagulation equation because the equation is rooted in the abstract size space as opposed to actual physical space. The difficulty lies in contradictory requirements imposed by sufficient spatial resolution and ability to neglect fluctuations, as discussed below.
(2) Even if a statistically homogeneous texture-less equilibrium drop size distribution were to be attained at some moment, the associated state of perfect spatial randomness would be unstable at short distances because of drop fragmentation. In other words, drop fragmentation turns the left panel of Fig. 1 into the middle one as detailed below.

Let us discuss the two comments, in turn. Consider the notion of a continuously varying size distribution function in the coagulation equation and ask about the physical meaning of $p = p(D, \mathbf{r}, t)$ at a specified position $\mathbf{r}$.[8] Clearly, the particle number (or expectation value) at a point is zero ($\Delta V = 0$) and in order to avoid the ''shot noise'' fluctuations in the expected drop number, one has to introduce a measurement volume ΔV sufficiently large to contain many raindrops. More precisely, for the success of continuous description, ΔV must be so large that the count fluctuations for *all* sizes can be neglected. But can we accomplish this and yet resolve spatial texture?

To test, we calculate the expected number of drops within a range ΔD at t and $\mathbf{r}$, with the control volume ΔV centered at $\mathbf{r}$. This is given by the product $(c\Delta V)p(D, \mathbf{r}, t)\Delta D$. In order to neglect fluctuations, in each size bin, we must at least satisfy

$$(c\Delta V)p(D, \mathbf{r}, t)\Delta D = Np(D, \mathbf{r}, t)\Delta D = N(D)\Delta D \gg 1. \tag{9}$$

To be more precise, one can obtain a lower (optimistic) bound by resorting to the Poisson distribution, and employ the ''$\sqrt{N}$'' rule: $[N(D)\Delta D]^{-1/2} = \epsilon$ where ϵ denotes desired accuracy (coefficient of variation of drop counts). Hence, the number of drops in every bin size must satisfy $N(D)\Delta D \geqslant 1/\epsilon^2$. This constraint is most stringent for the rarest largest drops. For example, in order to achieve ϵ of a few percent, using the density given by (8) with $\Delta D = 0.2$ mm, $\bar{D} = 0.6$ mm, $D = 3$ mm, and $c \sim 10^3\ \mathrm{m}^{-3}$ requires a measurement volume V on the order of several hundred cubic meters! Hence, we conclude that it is impossible to work with a statistically meaningful $p(D, \mathbf{r}, t)$ and yet resolve spatial correlations below the scale of a few meters.

Is there a way to avoid this difficulty? Perhaps one could get around this problem by resorting to an ensemble interpretation of $p(D, \mathbf{r}, t)$ (or expectation values) and invoking the ergodic hypothesis. This might allow a multitude of rain realizations with the same $p(D, \mathbf{r}, t)$. The first problem with this alternative is that any individual realization of such an ensemble would still be dominated by the discrete count fluctuation (''shot noise'') when small scales are considered. This has been discussed above. Furthermore, in our opinion, the ensemble alternative is not a viable one because spatial correlations occur in the real physical (rather than the abstract ensemble) space as is illustrated in the following example. Consider a deliberately extreme case of a collection of rain clouds, widely separated in *real*

[8] The coagulation equation is often written in terms of the number of particles of a given size $N(D, t, \mathbf{r})$. Our distribution function p is recovered by normalizing with the total particle number $N = cV$.

space. Furthermore, let each cloud contain raindrops of a single size but differ from cloud to cloud. In the ensemble space, the size distribution can still be defined by the relative number of such single drop size clouds in the entire ensemble but what is the physical meaning of such a distribution? No interaction occurs among different sizes because of the physical separation which renders the evolution of a size distribution predicted by the coagulation equation physically meaningless.

The conflicting requirements of a statistically meaningful drop size distribution, on one hand, and of sufficient spatial resolution on the other may be of practical significance in many applications. This difficulty may cause spurious radar reflectivity – rainfall ($Z - R$) relations (Jameson and Kostinski, 2002). Furthermore, as is well known, e.g., see Rinehart (1991, pp. 166–167), the very notion of the ''ground truth'' applied to disdrometers when used to ''validate'' radar-derived $Z - R$ relations can be misleading when the disdrometer measurement volumes are small.

Let us conclude by commenting on the question of rain texture stability with respect to the ''coalescence vs. breakup equilibrium'' size distribution. Above we concluded that spatial correlations are incompatible with the abstract ''size space'' point of view, enforced by the coagulation equation. On the other hand, lack of spatial correlations (as implied by the coagulation equation) defines perfect spatial randomness. So is the notion of the ''equilibrium'' size distribution compatible with perfect spatial randomness? The answer is no and it is an interesting argument.

Drop fragmentation (like birth) is spatially localized as the fragments are adjacent to each other right after drop break-up. This creates local ''bursts'' of concentration which coalescence cannot quickly counter through simple pair-wise collisions. An initially purely random (Poisson) field of drops, therefore, actually creates persistent super-Poisson droplet fluctuations of concentrations of smaller drop sizes (fragments). Schematically, the middle column pattern of Fig. 2 suddenly turns into the right column pattern. This phenomenon has recently been vividly illustrated by Young et al. (2001) in the context of reproductive pair correlations and clustering of organisms. In our case, such cluster production once again focuses attention on the scale dependence of the size-distribution.

Concluding remarks

As was pointed out in Lovejoy et al. (2003), few studies have been devoted to the question of stochastic scaling at really short distances (mm to m region). In response, here we propose a new definition of texture for rain which captures all scales in a single function (within stationary and homogeneous framework). We urge experimentalists to apply the new formalism to high resolution rain data (e.g., such as reported in Lovejoy et al. (2003)). Our proposed formalism is general unlike, for example, the fractal approach which assumes power-law behavior for the pair correlation function. We have also shown that fine texture (particularly negatively correlated rain) is not compatible with the idea of concentration fluctuations or inhomogeneities. Finally, we established rough lower bounds for length scales beyond which the integral parameters such as rain rate or size distribution become meaningful.

Uncited references

Jameson and Kostinski (2001), Kostinski and Shaw (2001).

Acknowledgements

We would thank one of the reviewers for pointing out the reference to Sasyo (1965). This work was supported by NSF Grants ATM01-0106271 (A.K. and M.L.), ATM02-42418 (A.J.), and the National Defense Science and Engineering Grant Program (M.L.).

References

Atlas, D., Ulbrich, C., 2000. An observationally based conceptual model of warm oceanic convective rain in the tropics. Journal of Applied Meteorology 39, 2165–2181.
Bras, R.L., Rodriguez-Iturbe, I., 1993. Random Functions and Hydrology. Dover.
Brenner, M., 1999. Screening mechanisms in sedimentation. Physics of Fluids 11 (4), 754–772.
Calder, I., 1996. Dependence of rainfall interception on drop size: 1. Development of the two-layer stochastic model. Journal of Hydrology 185, 363–378.
Calder, I., Hall, R., Rosier, P., Bastable, H., Prasana, K., 1996. Dependence of rainfall interception on drop size: 2. Experimental determination of the wetting functions and two-layer stochastic parameters for five tropical tree species. Journal of Hydrology 185, 379–388.
Chiu, C.-L. (Ed.), 1971. Stochastic Hydraulics. University of Pittsburgh Press, Pittsburgh, PA.
Clarke, R., 1998. Stochastic Processes for Water Scientists: Developments and Applications. Wiley, Great Britain.
Cox, D.R., Isham, V., 1980. Point Processes. Chapman and Hall, London.
Feller, W., 1966. An introduction to probability theory and its applications. An Introduction to Probability Theory and its Applications, vol. 2. Wiley, United States.
Friedlander, S., 2000. Smoke, Dust, and Haze: Fundamentals of Aerosol Dynamics, second ed. Oxford Press, New York.
Gupta, V.K., Waymire, E., 1990. Multiscaling properties of spatial rainfall and river flow distributions. Journal of Geophysical Research 98 (D3), 1999–2009.
Iranpour, R., Chacon, P., 1988. Basic Stochastic Processes: The Mark Kac Lectures, New York.
Jameson, A.R., Kostinski, A.B., 1999. Fluctuation properties of precipitation. Part IV: fine scale clustering of drops in variable rain. Journal of the Atmospheric Sciences 56 (1), 82–91.
Jameson, A.R., Kostinski, A.B., 2000. Fluctuation properties of precipitation. Part VI: observations of hyperfine clustering and drop size distribution structures in three-dimensional rain. Journal of the Atmospheric Sciences 57 (3), 373–388.
Jameson, A.R., Kostinski, A.B., 2001. What is a raindrop size distribution? Bulletin of the American Meteorological Society 82 (6), 1169–1177.
Jameson, A.R., Kostinski, A.B., 2002. Spurious power-law relations among rainfall and radar parameters. Quarterly Journal of the Royal Meteorological Society 128, 2045–2058.
Kostinski, A.B., Jameson, A.R., 1997. Fluctuation properties of precipitation. Part I: on deviations of single size drop counts

from the Poisson distribution. Journal of the Atmospheric Sciences 54 (17), 2174–2186.

Kostinski, A.B., Jameson, A.R., 1999. Fluctuation properties of precipitation. Part III: On the ubiquity and emergence of the exponential size spectra. Journal of the Atmospheric Sciences 56 (1), 111–121.

Kostinski, A.B., Jameson, A.R., 2000. On the spatial distribution of cloud particles. Journal of the Atmospheric Sciences 57 (7), 901–915.

Kostinski, A.B., Shaw, R.A., 2001. Scale-dependent droplet clustering in turbulent clouds. Journal of Fluid Mechanics 434, 389–398.

Landau, L., Lifshitz, L., 1980. Statistical Physics, third ed. Pergamon Press, New York.

Lovejoy, S., Lilley, M., Desaulniers-Soucy, N., Schertzer, D., 2003. Large particle number limit in rain. Physical Review E 68, 025301.

Lovejoy, S., Schertzer, D., 1995. Multifractals and rain. In: Kundzewicz, Z.W. (Ed.), New Uncertainty Concepts in Hydrology and Water Resources. Cambridge University Press, New York.

Marsan, D., Schertzer, D., Lovejoy, S., 1996. Causal space-time multifractal processes: Predictability and forecasting of rain fields. Journal of Geophysical Research 101 (D21), 26333–26346.

Marshall, J.S., Palmer, W.M., 1948. The distributions of raindrops with size. Journal of Meteorology 5, 165–166.

Ornstein, L., Zernike, F., 1914. Accidental deviations of density and opalescence at the critical point of a single substance. Proceedings of Academic Science 17, 793–806.

Peters, O., Hertlein, C., Christensen, K., 2002. A complexity view of rainfall. Physical Review Letters 88 (1), 018701-1–018701-4.

Pruppacher, H.R., Klett, J.D., 1997. Microphysics of Clouds and Precipitation, second ed. Kluwer Academic Publishers, Dordrecht.

Rinehart, R.E., 1991. Radar for Meteorologists, second ed. Knight Printing Company, Fargo, North Dakota.

Sasyo, Y., 1965. On the probabilistic analysis of precipitation particles. In: Proceedings of the International Conference on Cloud Physics, International Association of Meteorological and Atmospheric Physics.

Shaw, R.A., Kostinski, A.B., Larsen, M.L., 2002. Towards quantifying droplet clustering in clouds. Quarterly Journal of the Royal Meteorological Society 128, 1043–1057.

Spence, J., 2002. Spaced-out electrons. Nature 418, 377.

Srivastava, R., 1971. Size distribution of raindrops generated by their breakup and coalescence. Journal of the Atmospheric Sciences 28, 410–415.

Uijlenhoet, R., Porra, J., Sempere Torres, D., Creutin, J.-D. Analytical solutions to sampling effects in drop size distribution measurements during stationary rainfall: estimation of bulk rainfall variables. Journal of the Atmospheric Sciences (submitted).

Ulbrich, C., Atlas, D., 2002. On the separation of tropical convective and stratiform rains. Journal of Applied Meteorology 41, 188–195.

Valdez, M., Young, K., 1985. Number fluxes in equilibrium raindrop populations: A Markov chain analysis. Journal of the Atmospheric Sciences 42, 1024–1036.

Van Kampen, N., 1992. Stochastic Processes in Physics and Chemistry, Revised and Enlarged Edition. North-Holland Press, Amsterdam.

Vaneziano, D., Bras, R.L., Niemann, J.D., 1996. Nonlinearity and self-similarity of rainfall in time and a stochastic model. Journal of Geophysical Research 101 (D21), 26371–26392.

Vicsek, T., 1989. Fractal Growth Phenomena. World Scientific, Singapore.

Waymire, E., 1985. Scaling limits and self-similarity in precipitation fields. Water Resources Research 21 (8), 1271–1281.

Young, K., 1993. Microphysical Processes in Clouds. Oxford Univeristy Press, New York.

Young, W., Roberts, A., Stuhne, G., 2001. Reproductive pair correlations and the clustering of organisms. Nature 412, 328–331.

Zawadzki, I., 1995. Is rain fractal? In: Kundzewicz, Z.W. (Ed.), New Uncertainty Concepts in Hydrology and Water Resources. Cambridge University Press, New York.