Homework 3, PHYS 272 (Methods of Applied Physics)
Spring 2020
Due Friday, January 24th, 2020 at Beginning of Class

As always, turn your legible and complete answers in on separate paper. Remember, I can't give partial credit unless I can follow what you've done. Including words is usually a good thing for you.

1. Without a calculator or other technological aid, use series expansion to estimate the following to within 1% of the true value. (Since this is trivial to do with a calculator, you *must* show your work on these problems to earn credit.

   a) $\cos(\pi - .1)$ (the argument is in radians)
   b) $\sqrt{105}$
   c) $70^{1/6}$
   d) $\cos\left(\frac{5}{6}\right)$
   e) $\ln\left[100\left(\frac{e}{3}\right)^4\right]$
   f) $\ln(1.25)$

2. In PHYS 101/102 and 111/112, we often tell students that you can use the approximation $\sin x \approx x$ (in radians) for small values of x. Invariably, the question asked after that is "how small does x have to be to use that assumption?" Well, let's explore that a little bit. In truth, $\sin x$ can be eactly represented by the infinite series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots$$

   (I assume you already knew that). The smaller x is, the smaller x^n is as well, and the terms after the first one become vanishingly small for very small values of x. When $x < 1$, it is easy to see that each term is always smaller than the term preceding it, so a good "first-order" correction to $\sin x \approx x$ for $x < 1$ is $\sin x \approx x - \frac{x^3}{6}$. Let us define the relative change in using the first-order correction as $Q = \frac{x - \left[x - \frac{x^3}{3!}\right]}{\left(x - \frac{x^3}{3!}\right)}$

   (Q is the same as calculating the percent error of using the first term if you assume using the first two terms is the correct answer).

   Find (using a calculator if you must) the value of x so that Q is equal to (see next page):

a) 0.25 ($\sim$ 25% error in ignoring the second term in the expansion of $\sin x$)

b) 0.10 ($\sim$ 10% error)

c) 0.05 ($\sim$ 5% error)

d) 0.01 ($\sim$ 1% error)

e) 0.001 ($\sim$ 0.1% error)

3. We're now going to take a somewhat careful look at a relatively ugly integral. Both the integrand and the antiderivative (if you look it up) can be well approximated (near $x = 0$) by a Taylor series with just a couple terms. Carefully follow the instructions below and consider the following integral:

$$\int e^{ax} \sin(nx) \mathrm{d}x$$

with a and n both positive constants. This integral is rather ugly if you don't have the aid of some computer algebra system, but that doesn't mean we can't get a pretty good approximation. Often, a quick series substitution can be your best friend.

a) Use the power-law expansions for e^{ax} and $\sin(nx)$ near $x = 0$ to approximate the integrand as a cubic polynomial. (Retain all terms involving x^0, x^1, x^2, and x^3 in the integrand).

b) Integrate the cubic polynomial you obtained in part (a) to come up with an approximate expression for the integral above. Your answer should be a fourth-order polynomial in x.

c) Use your answer to part (b) to numerically estimate:

$$\int_0^{0.1} e^{5x} \sin(3x) \mathrm{d}x$$

Do not use a calculator to evaluate this! It should be just a plug and chug into your answer for part b using the limits $x_{\mathrm{min}} = 0$ and $x_{\mathrm{max}} = 0.1$ with $a = 5$ and $n = 3$. You probably will have to do some long division, though.

d) If you would use a computer algebra system or integral tables, you would find that:

$$\int e^{ax} \sin(nx) \mathrm{d}x = \frac{e^{ax}}{a^2 + n^2} \left(a \sin(nx) - n \cos(nx)\right)$$

(notice that this is written as an indefinite integral). Use this formula (and a calculator, if desired), to evaluate the definite integral in part (c). Compare to your answer calculated in part (c) from your power-law approximation to the integral.

e) Let's actually explore the power-series expansion of the true antiderivative of the function given in part (d): Use the power-law expansions for e^{5x}, $\sin(3x)$ and $\cos(3x)$ to approximate $\frac{e^{5x}}{34}\left(5\sin(3x) - 3\cos(3x)\right)$ as a third order polynomial. (Actually a fourth-order polynomial makes more sense given what I asked you to do above, but this is messy enough and the fourth order term this time doesn't have that major of an effect – so you only have to go up to third order. You are welcome).

f) Plug in $x_{\text{max}} = 0.1$ and $x_{\text{min}} = 0$ into this power-law approximated anti-derivative in part (e) to see how close this is to your answers to part (d) and part(c).

g) Briefly explain why the cubic polynomial approximation for the integrand you developed in part (a) would not be suitable to approximate the following integral:

$$\int_3^{\pi} e^{2x}\sin(6x)\mathrm{d}x$$

4. This one is deceptively tricky! Which is bigger, π^{e} or e^{π}? Give a clear reason for your answer. Remember, no calculators/*Mathematica*/etc can be used to justify your answer! Your reason needs to be based on some sort of manipulation and/or valid logical argument. Make your argument clear for me to understand and follow; I'm grading your reasoning more than your answer here since obviously anyone can just pick up a calculator and see.

5. You may have never run into this, but there is a beast called the "double-factorial" $n!!$; if you haven't seen it, you might see it in QM or a few other places. This is NOT $(n!)!$. Rather, $n!! \equiv n(n-2)(n-4)(n-6)\ldots(1)$. For example, $5!! = 5\cdot 3\cdot 1 = 15$. For even numbers, you stop at 2, so $8!! = 8\cdot 6\cdot 4\cdot 2 = 384$. For even n, there is a relationship that reads:

$$n!! = f(n)\left(\frac{n}{2}\right)!$$

Derive/determine an expression for $f(n)$.