Homework 8, PHYS 272 (Methods of Applied Physics)
Spring 2020
Due Friday, March 6th at Start of Class

As always, turn your legible and complete answers in on separate paper. Remember, I can't give partial credit unless I can follow what you've done. Including words is usually a good thing for you.

1. Write the following equations in index notation. This symbol $\widetilde{A}$ indicates a matrix. (All of the equations below should be well-defined).

   a) $\vec{f} = \vec{a} + \vec{b} \times \vec{c}$
   b) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{f} \cdot \vec{g})|\vec{h}|^2\vec{k}$
   c) $\widetilde{A}\vec{x} - (\vec{y} \times \vec{z}) = \vec{b}$
   d) $\vec{\nabla}\phi + \vec{\nabla} \times \vec{a} = (\vec{\nabla} \cdot \vec{b})\vec{c}$
   e) $(\nabla^2\phi)\vec{a} + \widetilde{B}\vec{c} = \vec{d} \times \vec{f}$
   f) $\frac{\partial^2 \vec{u}}{\partial t^2} = |\vec{v}|^2\nabla^2\vec{u}$
   g) $\frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$
   h) $\vec{u} + (\vec{a} \cdot \vec{b})\vec{v} = |\vec{a}|^2(\vec{b} \cdot \vec{v})\vec{a}$

2. Simplify the following expressions as much as possible:

   a) $\delta_{ij}\delta_{ij}$
   b) $\delta_{ij}\delta_{jk}\delta_{ki}$
   c) $\epsilon_{ijk}\epsilon_{mjk}$
   d) $\epsilon_{ijk}\epsilon_{ijk}$
   e) $\delta_{ij}\epsilon_{jkm}$

MORE ON BACK!

3. The Helmholtz theorem tells us that any vector $\vec{F}$ can be written as the negative gradient of a scalar field plus the curl of a vector field. i.e.:

$$\vec{F} = -\vec{\nabla}V + \vec{\nabla} \times \vec{E}$$

   a) Rewrite the above vector equation in index notation.
   b) Manipulate the expression in part (a) to show that $\vec{\nabla} \times \vec{F} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$. (This doesn't mean just referencing some source that argues that this is true; it means manipulating the index notation to clearly progress from the expression in part (a) to something equivalent to the above in index notation).

4. Use index notation to demonstrate the following:

   a) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$
   b) $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$
   c) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$
   d) $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}$

5. Use index notation to simplify / rewrite the following to the number of terms specified. (Let ϕ be an arbitrary scalar function and $\vec{a}$ be an arbitrary vector function):

   a) $\vec{\nabla} \times (\phi\vec{\nabla}\phi)$ (when simplified, this can be written as a single term).
   b) $\vec{\nabla} \cdot (\phi\vec{\nabla}\phi)$ (when rewritten, this should have two terms).
   c) $\vec{\nabla} \times (\phi\vec{a})$ (when rewritten, this should have two terms).

6. Let $\vec{r}(t) \cdot \vec{r}(t) = 1$. Differentiate both sides of the equation. Based on your result, what is the geometrical relationship between $\vec{r}$ and $\dot{\vec{r}}$ (the time derivative of $\vec{r}$)?

7. The force on a charge q moving with velocity $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ in a magnetic field $\vec{B}$ is $\vec{F} = q(\vec{v} \times \vec{B})$. Since $\vec{\nabla} \cdot \vec{B} = 0$, we can write $\vec{B}$ as $\vec{B} = \vec{\nabla} \times \vec{A}$ where $\vec{A}$ (called the vector potential) is a vector function of x, y, z, t. If the position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ of the charge q is a function of time t show that $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\partial\vec{A}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{A}$.