Homework 10, PHYS 272 (Methods of Applied Physics)
Spring 2020
Due Friday, April 3rd to my email inbox (LarsenML@cofc.edu) at noon.

We will be turning homework in electronically for the rest of the semester. If possible, try to get your work in pdf form, though individual .jpg files are acceptable. Please do not use google drive; just include all files as attachments to my email at LarsenML@cofc.edu; you can also CC my personal email address (LarsenML@gmail.com) if you are concerned about CofC's email servers stripping out your content.

Cell phone pictures of your work are ok – though I would ask you to make sure they are legible before sending them to me. Remember, I can't give partial credit unless I can follow what you've done. Including words is usually a good thing for you.

1. Take the following ordinary differential equation and associated initial conditions:

$$\begin{aligned} \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + 2\frac{\mathrm{d}z}{\mathrm{d}t} + 5z &= 10\cos t \\ z(t=0) &= 2 \\ \left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=0} &= 1 \end{aligned}$$

   One of the following four equations is a valid solution.

$$\begin{aligned} z &= \sin t + 2\cos t - 2e^{-t}\sin(2t) \\ z &= (5-6t)e^{t} - \sin t - 3\cos t \\ z &= 2\cos t + \sin t \\ z &= e^{3t} + 2e^{-2t}\sin t + 1 \\ z &= 2\cos t + \sin t - 2e^{-t}\cos(2t) \end{aligned}$$

   a) Figure out (via any means of your choice), which of the four proposed solutions is the valid solution to the given differential equation. Identify your solution, and describe/show the methodology you used. (There are many.) (Note – "my classmate told me" is not going to get credit here.) If you did work to figure it out, show it.

   b) If we use the same differential equations but the new initial conditions $z(t=0) = 0$ and $\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=0} = 3$, a different choice is a valid solution to the differential equation. Identify which one it is and describe/show the methodology you used. (Same caveats here as the last part).

2. A body at a temperature of T_i is placed in a room of unknown temperature. (The room temperature is taken to be constant for all time). If you didn't know this, the time rate of change of the temperature of a cooling body is proportional to the temperature difference between the body and its surroundings.

   a) Develop a differential equation governing $T(t)$ (the temperature as a function of time) for the body if the proportionality constant is taken to be k and the temperature of the room is $T_\circ$. Make sure your differential equation makes sense so that when k is positive, an object heats up if its surroundings are at a higher temperature and cools down of its surroundings are at a colder temperature.

   b) Solve the differential equation you developed in part (a).

   c) Verify that your solution in part (b) above is consistent with the physically clear facts that as $t \to \infty$, $T(t) \to T_\circ$ and as $t \to 0$, $T(t) \to T_i$.

   d) Verify your solution in part (b) by plugging it back into the differential equation in part (a) and show both sides are equal.

3. Let x be a function of t.

   a) If the acceleration $a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \equiv \ddot{x} = A\sin(\omega t)$ for some constants A and ω, what is $x(t)$?

   b) The above solution should have two arbitrary constants besides A and ω. If $\dot{x}$ at $t = 0$ is 0 and x at $t = 0$ is -3, what is the solution for $x(t)$?

4. The momentum p of an electron at speed v near the speed of light c increases via the formula $p = \gamma m v = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ where m is a constant (the mass of the electron). If an electron is subject to a constant force F, Newton's second law describing its motion is:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right) = F$$

   a) If you solve this differential equation, you should find that the answer for $v(t)$ takes the form:

$$v(t) = \frac{c(Ft + A)}{g(A, t, F, m, c)}$$

   Where $g(A, t, F, m, c)$ is some function of variables t, F, m, c and the same integration constant that appears in the numerator. Find $g(A, t, F, m, c)$. Hint: what is $\int \mathrm{d}(\text{mess})$?

   b) Show that $v(t) \to c$ as $t \to \infty$.

   c) Find the distance traveled by the electron in time t if it starts from rest. *Do not use Mathematica or other computer software/assistance!* The following (indefinite) integral may come in handy: $\int \frac{t\mathrm{d}t}{(a^2+t^2)^{1/2}} = (a^2 + t^2)^{1/2}$. ($a$ is any constant). Choose your integration constant so that $x(t = 0) = 0$.

   d) Show that your expression for part c reduces to the expected/classical value $\Delta x = \frac{1}{2}\left(\frac{F}{m}\right)t^2$ for small t. (Hint: Binomial expansion!)

5. The differential equation for a particle moving vertically under the presence of air-resistance can be written as:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - \beta v$$

With β some constant associated with the physical properties of the system.

a) What is the steady-state value (e.g. free-fall speed) of this system? (Hint...what does steady-state mean in terms of v?)

b) If a particle of mass m is dropped from rest, find $v(t)$. (In other words, solve this differential equation under the initial condition $v(t = 0) = 0$). A change of variable may be helpful.

6. Given the fact that there are only a few weeks left to the semester, we are going to have to make some strategic choices about what additional content to cover. We still have some differential equations content we *must* cover, but after that we have a little bit of flexibility. Please rank your personal preferences regarding which topics you'd like us to cover in the remainder of the semester from 1 (least interested) to 4 (most interested); based on your preferences I will put together a schedule for the rest of the semester that makes the most sense.

a) More differential equations. (Differential equations are used all over Physics, and typically a good undergraduate sequence in Differential Equations takes a full year. Spending more time on different differential equations could be of some benefit.)

b) Probability (Near and dear to my own heart, many students don't ever really develop a sufficiently deep understanding of probability. Students often are confounded by the seeming paradoxes that a careful study of probability can bring about. This content is most important in a Physics curriculum in Thermodynamics/Statistical Mechanics – but also has applications in Quantum Mechanics and Pops up in other somewhat surprising places).

c) Statistics (A common concern among faculty in our department is that students going through our curriculum never really get an exposure to statists. This topic has relevance to experimental design, data analysis, and thermodynamics/statistical mechanics. It is used surprisingly often, and MATH 250 often doesn't cut it. Pairing statistics with probability is common, but not necessarily required.)

d) Fourier Transforms/Integral Transforms/Convolutions (An important topic for understanding how to decompose functions into their constituent periodic functions. The study of Fourier Transforms is key to a lot of experimental and data analysis methods in Physics, and – depending on the specific sub-topics that are emphasized – can give additional insight into solving differential equations, acoustics, solid state physics, signal analysis, waves/optics, and literally dozens of other areas. It is hard to come up with homework problems for this area, so a treatment of this topic might be primarily conceptual – which might make it harder to prepare for questions on this topic for the final exam).

7. Do you have any comments about any of the above listed topics that you'd like me to take into account as I tabulate your results? (A compelling argument for or against a topic might be used as a tie-breaker for the class vote).