An example of an analytically solvable coupled Physical system

In class, we just finished solving equations of motion under the conditions of linear and quadratic air resistance. We were only able to come up with a closed-form solution for quadratic air resistance in the $\hat{y}$ or $\hat{x}$ directions, and one theory as to why is that those are the only two directions we can choose (assuming we have lined up our coordinate system so that $\hat{y}$ is up) where the equations of motion become uncoupled. One might then conclude that any coupled system is impossible to solve; this is a counter-example – motion of a charged particle in uniform $\vec{B}$ field. Your textbook comes up with the same final answer I do here, but does so in a different way.

We start by talking about a particle of charge q in the presence of the following electric and magnetic fields:

$$\vec{E} = \vec{0}$$
$$\vec{B} = B\hat{z}$$

where B is the (constant) magnitude of the magnetic field $\vec{B}$. The initial velocity of the particle is not explicitly specified, so we merely write:

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$$
$$\vec{v}_\circ = v_{x\circ}\hat{x} + v_{y\circ}\hat{y} + v_{z\circ}\hat{z}$$

Since the Lorentz force can be written:

$$\vec{F}_{EM} = q(\vec{E} + \vec{v} \times \vec{B})$$

we can find the time-dependent force by computing:

$$\vec{F}_{EM} = 0 + q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = q(v_yB\hat{x} - v_xB\hat{y})$$

Thus, appealing to Newton's second law to develop equations of motions for each of the 3 cartesian directions, we have:

$$
\begin{aligned}
F_x = m\dot{v}_x &= qv_yB \\
F_y = m\dot{v}_y &= -qv_xB \\
F_z = m\dot{v}_z &= 0
\end{aligned}
$$

Clearly, the z-coordinate is easiest to deal with, and we have:

$$
\begin{aligned}
m\frac{\mathrm{d}v_z}{\mathrm{d}t} &= 0 \\
m\int_{v_{z\circ}}^{v_z(t)} \mathrm{d}v_z &= 0 \\
m[v_z(t) - v_{z\circ}] &= 0 \\
\boxed{v_z(t) = v_{z\circ}}
\end{aligned}
$$

Which isn't really a surprise. With a magnetic field in the z direction, the only thing we know is that there is never a force in the z direction and, hence, the z velocity stays the same forever.

The x and y components above, however, are coupled – which so far has meant trouble for us. The book uses complex numbers to solve this – which is interesting. For those of you who are less comfortable with i, however, you can use standard diffeq type techniques to find $v_x(t)$ and $v_y(t)$ (and, ultimately, $x(t)$ and $y(t)$ if you know initial conditions).

We start by looking at the first expression:

$$
\begin{aligned}
m\dot{v}_x &= qv_yB \\
\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{v}_x) &= \frac{\mathrm{d}}{\mathrm{d}t}(qv_yB) \\
\dot{m}\dot{v}_x + m\ddot{v}_x &= \dot{q}v_yB + q\dot{v}_yB + qv_y\dot{B}
\end{aligned}
$$

Noting that m, q, and B are assumed constant (and thus their time derivatives are zero) we then have:

$$\begin{aligned} m\ddot{v}_x &= q\dot{v}_y B \\ \dot{v}_y &= \frac{m}{qB}\ddot{v}_x \end{aligned}$$

We will use this expression for $\dot{v}_y$ and plug it into the second expression (which we obtained by looking at the forces in the y-direction):

$$\begin{aligned} m\dot{v}_y &= -qv_x B \\ m\left(\frac{m}{qB}\ddot{v}_x\right) &= -qv_x B \\ \ddot{v}_x + \left(\frac{qB}{m}\right)^2 v_x &= 0 \end{aligned}$$

This is a differential equation that should look familiar. IF YOU DO NOT KNOW THIS ALREADY, YOU SHOULD! The solution of the second-order differential equation $\ddot{x} + \kappa^2 x = 0$ has a solution given by $x(t) = A\sin(\kappa t) + B\cos(\kappa t)$. If you are only going to know a few diffeq solutions, this is one of them you should have handy. Thus, we can write:

$$v_x(t) = \alpha\cos\left(\frac{qB}{m}t\right) + \beta\sin\left(\frac{qB}{m}t\right)$$

Now we apply initial conditions and note that since $v_x(t=0) = v_{x\circ}$, we can set $\alpha = v_{x\circ}$ and we end up with the expression:

$$v_x(t) = v_{x\circ}\cos\left(\frac{qBt}{m}\right) + \beta\sin\left(\frac{qBt}{m}\right)$$

We now can go back and plug THIS back into the initial expression for the x motion and obtain:

$$
\begin{aligned}
m\dot{v}_x &= qBv_y \\
m\left[\frac{-qBv_{x\circ}}{m}\sin\left(\frac{qBt}{m}\right) + \frac{qB\beta}{m}\cos\left(\frac{qBt}{m}\right)\right] &= qBv_y \\
v_y = \frac{m}{qB}\left[\frac{-qBv_{x\circ}}{m}\sin\left(\frac{qBt}{m}\right) + \frac{qB\beta}{m}\cos\left(\frac{qBt}{m}\right)\right] \\
v_y = -v_{x\circ}\sin\left(\frac{qBt}{m}\right) + \beta\left(\frac{qBt}{m}\right)
\end{aligned}
$$

Applying the other initial condition that $v_y(t=0) = v_{y\circ} = \beta$, we have:

$$\boxed{v_x(t) = v_{x\circ}\cos(\omega t) + v_{y\circ}\sin(\omega t)}$$
$$\boxed{v_y(t) = v_{y\circ}\cos(\omega t) + v_{x\circ}\sin(\omega t)}$$

with $\omega \equiv \frac{qB}{m}$. Note that the magnitude of the total velocity vector is constant (e.g. $v_x^2 + v_y^2 + v_z^2 = v_{x\circ}^2 + v_{y\circ}^2 + v_{z\circ}^2$), so the magnetic field is doing no work.

What was the point to this exercise? We merely showed that when the coordinates are coupled, all hope is necessarily not lost. We were still able to solve the differential equations to find $\vec{v}(t)$ (and we could easily integrate the expressions to find $\vec{r}(t)$, even in this case where the x-forces depend on the y-velocity and the y-forces depend on the x-velocity. We are only in trouble SOMETIMES when things are coupled. In short, we need to get lucky.

(For what it is worth, if $v_{z\circ} = 0$, this system results in uniform circular motion. If $v_{z\circ} \neq 0$, we end up with helical motion with the axis of the helix either parallel or antiparallel to $\hat{z}$).