Force Vectors in Electrostatics – A simple example

Assume there are two charges as follows:

- q_1, a charge of 83.2 nC at position $\langle 3.1 \text{ cm}, 2.4 \text{ cm}\rangle$
- q_2, a charge of -1.35 μC at position $\langle -1.2 \text{ cm}, 5.3 \text{ cm}\rangle$

Charge q_1 is fixed (unable to move). What is the force on charge q_2 due to q_1?

This isn't very hard if all we are asked to do is find the magnitude, then we could just use Coulomb's law:

$$|\vec{F}_{q_1 q_2}| = \frac{k|q_1||q_2|}{r_{12}^2}$$

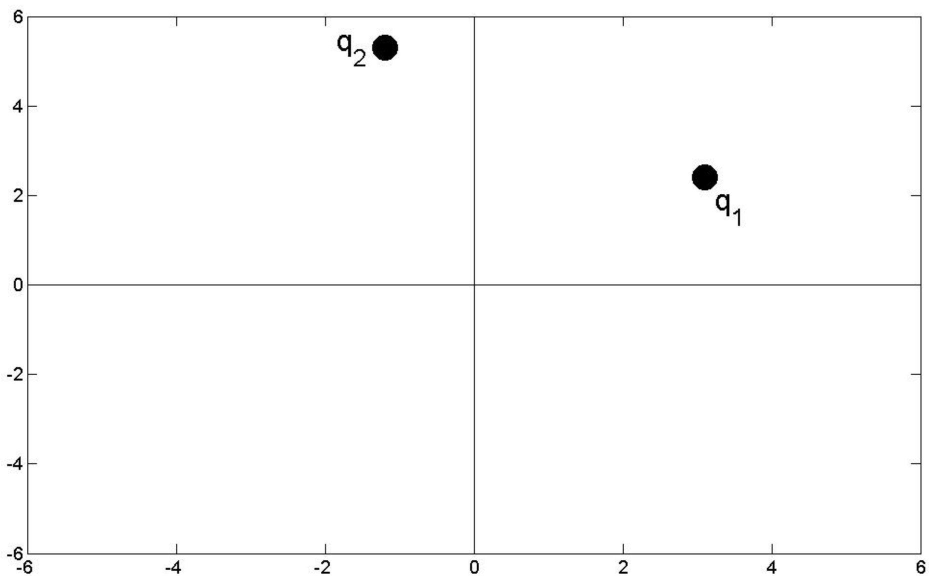
plug in the appropriate values and we're done. However, we're asked to find the force – which is a vector. This means we'll need a bit more work.

Here's our plan of attack:

- Look at the positions of the charges to figure out if the force on q_2 due to q_1 is in the $+x$ or $-x$ direction and the $+y$ or $-y$ direction.
- Calculate the magnitude of the total force on q_2 due to q_1 via Coulomb's law.
- Use some basic trigonometry to figure out what fraction of the force is in the x and y directions.
- Calculate the components of the total force $(\vec{F}_{q_1 q_2})_x$ and $(\vec{F}_{q_1 q_2})_y$.
- Use the component information just found to determine the final answer.

Directions of the force

Following our plan of attack above, we need to first figure out which direction the force on q_2 will be. A quick sketch might help us out a bit.



Now, since q_1 and q_2 have opposite signs, the force will be attractive. q_1 will thus pull q_2 in the positive x direction and the negative y direction. This will help us later on.

Total magnitude of the force

As noted above, we can write the total magnitude of the force via Coulomb's law:

$$|\vec{F}_{q_1 q_2}| = \frac{k|q_1||q_2|}{r_{12}^2}$$

We have to make sure to use proper units, however. We're given charges in μC and nC, and distances in cm. If we want to actually get an answer in Newtons, we need to use SI units for everything else – which means converting the charges to Coulombs (C) and the distances to meters. Recalling there are 100 cm in a meter, we can write:

$$|\vec{F}_{q_1 q_2}| = \frac{8.99 \times 10^9 \frac{\mathrm{N\ m}^2}{\mathrm{C}^2} \cdot (83.2 \times 10^{-9}\ \mathrm{C}) \cdot (1.35 \times 10^{-6}\ \mathrm{C})}{\left[((-.012\ \mathrm{m} - .031\ \mathrm{m})^2 + (.053\ \mathrm{m} - .024\ \mathrm{m})^2)^{1/2}\right]^2}$$
$$|\vec{F}_{q_1 q_2}| = \frac{8.99 \times 10^9 \frac{\mathrm{N\ m}^2}{\mathrm{C}^2} \cdot (83.2 \times 10^{-9}\ \mathrm{C}) \cdot (1.35 \times 10^{-6}\ \mathrm{C})}{(-.043\ \mathrm{m})^2 + (.029\ \mathrm{m})^2}$$

A couple things to note: first, we calculate r_{12} via taking the differences in the x components of the positions, squaring them, and adding to the differences in the y components and squaring them. The magnitude of that distance is the square root of this sum of squares (it is really the Pythagorean theorem hidden). However, we then have to square this at the end since we want r_{12}^2, so all we really need to do is to take $(\vec{r}_{12})_x^2 + (\vec{r}_{12})_y^2$.

When we calculate the total force, we come up with:

$$|\vec{F}_{q_1 q_2}| \approx 0.3754\ \mathrm{N}$$

So now we have the total force as well as the general direction – there is a component in the $+x$ direction and a component in the $-y$ direction as per our considerations above.

Magnitudes of the X and Y components

The electrostatic force – just like gravity – acts in the line connecting the two objects in question. On Earth, with gravity, we generally don't have to think about components of the force very often because we usually choose a coordinate system with gravity "down", so $\vec{g} = \langle 0, -9.81\ \mathrm{m/s}^2 \rangle$. (The entire force is in the y component.) Here, we have to resolve the force into its two basic components. Because it is parallel to the vector between them, we can write:

$$(\vec{F}_{q1q2})_x = |\vec{F}_{q1q2}|\frac{(\vec{r}_{12})_x}{r_{12}}$$
$$(\vec{F}_{q1q2})_y = |\vec{F}_{q1q2}|\frac{(\vec{r}_{12})_y}{r_{12}}$$

where $(\vec{r}_{12})_x$ corresponds to the x component of the vector pointing from q_1 to q_2, $(\vec{r}_{12})_y$ corresponds to the y component of the vector pointing from q_1 to q_2, and r_{12} is the total magnitude of the vector pointing from q_1 to q_2 (aka the distance between the two charges).

(Note, the reason we've troubled ourselves to find out what the final direction of the force is as step 1 is so that we don't have to be too careful about if we really need "the vector from q_1 to q_2 or the vector from q_2 to q_1. This technique will give us the magnitude, and then we slap on a negative sign if necessary based on our physical reasoning from the beginning of the problem).

Following this logic, then, we have:

$$(\vec{F}_{q1q2})_x = 0.3754\text{ N}\frac{(-.012\text{ m}) - (.031\text{ m})}{((-.043\text{ m})^2 + (.029\text{ m})^2)^{1/2}} \approx -0.3112\text{ N}$$
$$(\vec{F}_{q1q2})_y = 0.3754\text{ N}\frac{(.053\text{ m}) - (.024\text{ m})}{((-.043\text{ m})^2 + (.029\text{ m})^2)^{1/2}} \approx 0.2099\text{ N}$$

Now, we're not going to worry too much about the sign here since we already know what sign each component is supposed to have. We just did this step to find out what the magnitude of each component should be.

Values of the X and Y components

In the first section, we argued that this force should be in the positive x direction and the negative y direction, therefore our total force is:

$$\vec{F}_{q_1 q_2} = \langle .3112 \text{ N}, -.2099 \text{ N} \rangle$$

Depending on the problem statement, this can be our answer. However, often times people prefer to specify vectors with an angle and a direction instead of as two components. Let's do that as well.....

Final Answer

The final answer can either be what we just calculated:

$$\vec{F}_{q_1 q_2} = \langle .3112 \text{ N}, -.2099 \text{ N} \rangle$$

or we can write that it is a force of magnitude 0.3754 N. To completely specify the vector, however, we also must give a direction. Since the force is in the $+x$ direction and the $-y$ direction, we can specify the direction in terms of an angle with respect to the horizontal. If we form a triangle, then using basic trigonometry we can conclude that the tangent of the angle of declination (the tangent of the angle lower than the horizontal) is given by $|F|_y/|F|_x$, or:

$$\begin{aligned} \tan\theta &= \frac{.2099}{.3112} \\ \theta = \tan^{-1} &\frac{.2099}{.3112} \\ \theta &\sim 34^\circ \end{aligned}$$

So another way of saying our answer is to say that the total force is 0.3754 N at an angle 34° declined from the horizontal.

Force Superposition

Some worked-out examples of force superposition can be found in your text (see 19-2 and 19-3; read 19-3 carefully).

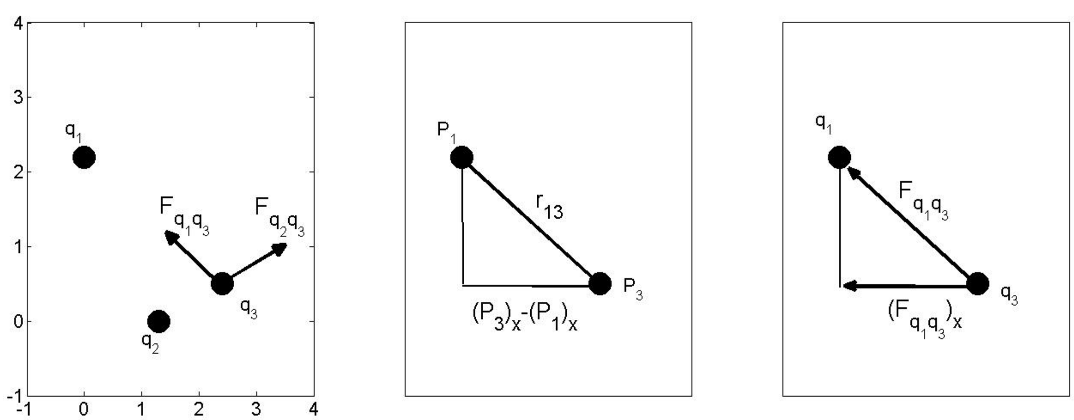
Here's another example.

Assume there are three charges as follows:

- q_1, a charge of $+4.7\ \mu$C at position $\langle 0, 2.2\ \text{m}\rangle$
- q_2, a charge of $-2.3\ \mu$C at position $\langle 1.3\ \text{m}, 0\rangle$
- q_3, a charge of $-3.1\ \mu$C at position $\langle 2.4\ \text{m}, 0.5\ \text{m}\rangle$

Charges q_1 and q_2 are unable to move. What is the total force on charge q_3 due to these other two charges?

I always find it easiest to start with a picture, so let's plot the alignment of the three charges. While we're at it, I'll plot a couple more things that might come in handy later.



The left figure shows the position of the three charges, and I've already figured out the directions of the two forces acting on q_3, noting that the force between q_1 and q_3 must be attractive (since they have different signs), and the force between q_2 and q_3 must be repulsive (since they have like charges).

We know from Coulomb's law that:

$$
\begin{aligned}
|\vec{F}_{q_1q_3}| &= \frac{k|q_1||q_3|}{r_{13}^2} \\
|\vec{F}_{q_2q_3}| &= \frac{k|q_2||q_3|}{r_{23}^2}
\end{aligned}
$$

Also, for notational convenience, let

- q_1 be at point P_1 with co-ordinates $(P_1)_x$ and $(P_1)_y$
- q_2 be at point P_2 with co-ordinates $(P_2)_x$ and $(P_2)_y$
- q_3 be at point P_3 with co-ordinates $(P_3)_x$ and $(P_3)_y$

The middle and right parts of the included figure help to realize that some basic geometry can help us out in resolving the x and y coordinates of these forces. The figures are drawn for $\vec{F}_{q_1q_3}$. The middle figure shows how the vector $\vec{r}_{13}$ can be found by looking at the x and y components of the differences in the position vectors for P_1 and P_3. The right figure shows how we can decompose the force vector $\vec{F}_{q_1q_3}$ into its x and y coordinates. The key observation is that these two figures show similar triangles; all the interior angles are identical, because the force is in the same direction as the separation vector. Therefore, the ratio of the "bottom" to the "hypotenuse" in both triangles is the same. As an equation (and working just with the magnitudes of the vectors), this means we can write:

$$
\begin{aligned}
\frac{|(\vec{F}_{q_1q_3})_x|}{|\vec{F}_{q_1q_3}|} &= \frac{(P_3)_x - (P_1)_x}{r_{13}} \\
|(\vec{F}_{q_1q_3})_x| = |\vec{F}_{q_1q_3}| &\cdot \frac{(P_3)_x - (P_1)_x}{r_{13}}
\end{aligned}
$$

Using the exact same reasoning, we can write both components of both forces:

$$
\begin{aligned}
|(\vec{F}_{q_1q_3})_x| &= |\vec{F}_{q_1q_3}| \cdot \frac{(P_3)_x - (P_1)_x}{r_{13}} \\
|(\vec{F}_{q_1q_3})_y| &= |\vec{F}_{q_1q_3}| \cdot \frac{(P_1)_y - (P_3)_y}{r_{13}} \\
|(\vec{F}_{q_2q_3})_x| &= |\vec{F}_{q_2q_3}| \cdot \frac{(P_3)_x - (P_2)_x}{r_{23}} \\
|(\vec{F}_{q_2q_3})_y| &= |\vec{F}_{q_2q_3}| \cdot \frac{(P_3)_y - (P_2)_y}{r_{23}}
\end{aligned}
$$

Since we know $\vec{F}_{q_1q_3}$ is up and to the left, we know the x component of this force must be negative and the y component must be positive. Since we know $\vec{F}_{q_2q_3}$ is up and to the right, we know the x and y components of this force must both be positive.

Now, as soon as we are able to write r_{13} and r_{23} in a convenient form, we're pretty much done. Using the Pythagorean theorem:

$$
\begin{aligned}
r_{13} &= \left[((P_1)_x - (P_3)_x)^2 + ((P_1)_y - (P_3)_y)^2\right]^{1/2} \\
r_{23} &= \left[((P_2)_x - (P_3)_x)^2 + ((P_2)_y - (P_3)_y)^2\right]^{1/2}
\end{aligned}
$$

Putting everything together, we can write:

$$
\begin{aligned}
(\vec{F})_x &= \frac{-k|q_1||q_3|}{r_{13}^3}\left((P_3)_x - (P_1)_x\right) + \frac{k|q_2||q_3|}{r_{23}^3}\left((P_3)_x - (P_2)_x\right) \\
(\vec{F})_y &= \frac{k|q_1||q_3|}{r_{13}^3}\left((P_1)_y - (P_3)_y\right) + \frac{k|q_2||q_3|}{r_{23}^3}\left((P_3)_y - (P_2)_y\right)
\end{aligned}
$$

Plugging everything is reasonably tedious and it is easy to make a mistake. However, if you do it right, you should get:

$$(\vec{F})_x = 0.0276 \text{ N}$$
$$(\vec{F})_y = 0.0269 \text{ N}$$

Thus, the total force is:

$$|\vec{F}| = \left[(\vec{F})_x^2 + (\vec{F})_y^2 \right]^{1/2} = 0.0385 \text{ N}$$

Of course, this is only part of the answer. This gives us the total magnitude, but not the direction. Noting that the resulting force has a positive x and y component, we can use trigonometry to infer that:

$$\theta = \tan^{-1} \left(\frac{(\vec{F})_y}{(\vec{F})_x} \right) \approx 44.2^\circ \text{ inclined from } x \text{ direction}$$