Assignment II, PHYS 272 (MAP)
Fall 2014
Due 8/29/14 at start of class

Note! Unless specified otherwise in the question, it is expected your solutions do NOT rely on the use of *Mathematica*, MATLAB, Maple, etc. *nor* calculators, slide-rules, integral tables, internet sources, etc! The idea is that you should be able to do these things *without* outside resources!

1. An important function in Solid-State Physics is the Lennard-Jones potential. This relationship takes on the following form:

$$f(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

with A and B positive real constants, and r indicating the distance form the origin (thus only positive real values of r are necessary).

   a) This function has one global minimum. Find the value of r at this minimum.
   b) What is the smallest possible value of $f(r)$?
   c) Sketch a plot of this function (by hand). Make sure to clearly indicate any important values.

2. Sketch a plot of the following function (by hand). Make sure to clearly indicate any important values. Assume a is a positive real constant.

$$f(x) = \frac{(x-a)}{a^2 - (x-a)^2}$$

3. Approximate, to within 1% accuracy (by hand!):

   a) $70^{1/6}$
   b) $\cos\left(\frac{5}{6}\right)$
   c) $\ln\left[100\left(\frac{e}{3}\right)^4\right]$

4. In introductory Physics, we often tell students that you can use the approximation $\sin x \approx x$ (in radians, of course) for small values of x. Invariably, students ask "how small does x have to be to use this approximation?" Let's explore this a bit. A full representation of $\sin x$ can be written as:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots$$

(I assume you already knew that, but just to be safe I'm giving it to you). The smaller x is, the smaller x^n is as well, and the terms after the first one become vanishingly small for small values of x. When $x < 1$, you can pretty clearly demonstrate that each successive term is always smaller than the term preceding it, so a good "first-order" correction to $\sin x \approx x$ for $x < 1$ is $\sin x \approx x - \frac{x^3}{6}$. Let us define the relative change in using this correction by exploring the quantity:

$$Q = \frac{x - \left[x - \frac{x^3}{3!}\right]}{\left(x - \frac{x^3}{3!}\right)}$$

Find (using a calculator, if you must), the value of x so that Q is equal to:

a) 0.25($\sim$ 25% error in ignoring the second term).
b) 0.10($\sim$ 10% error)
c) 0.05($\sim$ 5% error)
d) 0.01($\sim$ 1% error)
e) 0.001($\sim$ 0.1% error)

5. We're now going to take a somewhat careful look at a relatively ugly integral. Both the integrand and the antiderivative (if you look it up) can be well approximated (near $x = 0$) by a power-law series with just a couple terms. Carefully follow the instructions below and consider the following integral:

$$\int e^{-ax} \sin(nx) \mathrm{d}x$$

with a and n both positive constants. This integral is rather ugly if you don't have the aid of some computer algebra system, but that doesn't mean we can't get a pretty good approximation. Often, a quick power-law substitution can be your best friend.

a) Use the power-law expansions for e^{-ax} and $\sin(nx)$ near $x = 0$ to approximate the integrand as a cubic polynomial. (Retain all terms involving x^0, x^1, x^2, and x^3 in the integrand).

b) Integrate the cubic polynomial you obtained in part (a) to come up with an approximate expression for the integral above. Your answer should be a fourth-order polynomial in x.

c) Use your answer to part (b) to numerically estimate:

$$\int_0^{0.1} e^{-3x}\sin(5x)\mathrm{d}x$$

(You may use a calculator, if you wish, to evaluate this. It should be just a plug and chug into your answer for part *b* using the limits $x_{\text{min}} = 0$ and $x_{\text{max}} = 0.1$ with $a = 3$ and $n = 5$.

d) If you use *Mathematica* or integral tables, you should find that:

$$\int e^{mx}\sin(px)\mathrm{d}x = \frac{e^{mx}}{m^2+p^2}\left(m\sin(px) - p\cos(px)\right)$$

(notice that this is written as an indefinite integral). Use this formula (and a calculator, if desired), to evaluate the definite integral in part (c). Compare to your answer calculated in part (c) from your power-law approximation to the integral.

e) Let's actually explore the power-series expansion of the true antiderivative of the function given in part (d): Use the power-law expansions for e^{-3x}, $\sin(5x)$ and $\cos(5x)$ to approximate $\frac{e^{-5x}}{34}\left(-3\sin(5x) - 5\cos(5x)\right)$ as a third order polynomial.

f) Plug in $x_{\text{max}} = 0.1$ and $x_{\text{min}} = 0$ into this power-law approximated anti-derivative in part (e) to see how close this is to your answers to part (d) and part(c).

6. You may have never run into this, but there is a beast called the "double-factorial" $n!!$; if you haven't seen it, you might see it in QM or a few other places (it often creeps up in PDEs). $n!!$ is NOT $(n!)!$. Rather, $n!! \equiv n(n-2)(n-4)(n-6)\ldots(1)$. For example, $5!! = 5\cdot 3\cdot 1 = 15$. For even numbers, you stop at 2, so $8!! = 8\cdot 6\cdot 4\cdot 2 = 384$. For even n, there is a relationship that reads:

$$n!! = f(n)\left(\frac{n}{2}\right)!$$

find $f(n)$.

7. This one is deceptively tricky! Which is bigger, π^{e} or e^{π}? Give a clear reason for your answer. Remember, no calculators/*Mathematica*/etc! Your reason needs to be based on some sort of manipulation and/or valid logical argument.