Homework 2, PHYS 415 (Fluid Mechanics)
Spring 2019
Due Thursday 17th January 2019 at Beginning of Class

As always, turn in your legible and annotated work on separate paper.

Let:

$$\vec{a} = yz\hat{x} + xz\hat{y} + xy\hat{z}$$
$$\vec{b} = x^2e^{-z}\hat{x} + y^3\ln(x)\hat{y} + z\cosh(iy)\hat{z}$$
$$\vec{c} = r^2\hat{r} \qquad \text{(spherical coordinates)}$$
$$\vec{d} = \frac{1}{s^2}\hat{s} \qquad \text{(cylindrical coordinates)}$$
$$f = 3x^2yz$$
$$g = \cosh(xy)$$
$$h = r^3 \qquad \text{(spherical coordinates)}$$

1. Compute the following:
   a) $\vec{\nabla} \cdot \vec{a}$
   b) $\vec{\nabla} \cdot \vec{b}$
   c) $\vec{\nabla} \cdot \vec{c}$
   d) $\vec{\nabla} \cdot \vec{d}$
   e) $\vec{\nabla} \times \vec{a}$
   f) $\vec{\nabla} \times \vec{b}$
   g) $\vec{\nabla} \times \vec{c}$
   h) $\vec{\nabla} \times \vec{d}$
   i) $\vec{\nabla} f$
   j) $\vec{\nabla} g$
   k) $\vec{\nabla} h$
   l) $\nabla^2 f$ (if you haven't seen ∇^2 before, this means $\vec{\nabla} \cdot (\vec{\nabla} f)$).
   m) $\nabla^2 g$
   n) $\nabla^2 h$

2. Write the following equations in index notation. The tilde symbol above a variable (e.g. $\widetilde{A}$) indicates a matrix. (All of the equations below should be well-defined).

   a) $\vec{f} = \vec{a} + \vec{b} \times \vec{c}$
   b) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{f} \cdot \vec{g})|\vec{h}|^2\vec{k}$
   c) $\widetilde{A}\vec{x} - (\vec{y} \times \vec{z}) = \vec{b}$
   d) $\vec{\nabla}\phi + \vec{\nabla} \times \vec{a} = (\vec{\nabla} \cdot \vec{b})\vec{c}$
   e) $(\nabla^2\phi)\vec{a} + \widetilde{B}\vec{c} = \vec{d} \times \vec{f}$
   f) $\frac{\partial^2\vec{u}}{\partial t^2} = |\vec{v}|^2\nabla^2\vec{u}$
   g) $\frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$
   h) $\vec{u} + (\vec{a} \cdot \vec{b})\vec{v} = |\vec{a}|^2(\vec{b} \cdot \vec{v})\vec{a}$

3. Use index notation to demonstrate the following:

   a) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$
   b) $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$
   c) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$
   d) $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2\vec{a}$

4. The Helmholtz theorem tells us that any vector $\vec{F}$ can be written as the negative gradient of a scalar field plus the curl of a vector field. i.e.:

$$\vec{F} = -\vec{\nabla}V + \vec{\nabla} \times \vec{E}$$

   a) Rewrite the above vector equation in index notation.
   b) Manipulate the expression in part (a) to show that $\vec{\nabla} \times \vec{F} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E}$. (This doesn't mean just referencing some source that argues that this is true; it means manipulating the index notation to clearly progress from the expression in part (a) to something equivalent to the above in index notation).

5. Use index notation to simplify / rewrite the following to the number of terms specified. (Let ϕ be an arbitrary scalar function and $\vec{a}$ be an arbitrary vector function):

   a) $\vec{\nabla} \times (\phi\vec{\nabla}\phi)$ (when simplified, this can be written as a single term).
   b) $\vec{\nabla} \cdot (\phi\vec{\nabla}\phi)$ (when rewritten, this should have two terms).
   c) $\vec{\nabla} \times (\phi\vec{a})$ (when rewritten, this should have two terms).

6. Simplify the following expressions:

   a) $\delta_{ij}\delta_{ij}$

   b) $\delta_{ij}\delta_{jk}\delta_{ki}$

   c) $\epsilon_{ijk}\epsilon_{mjk}$

   d) $\epsilon_{ijk}\epsilon_{ijk}$

   e) $\delta_{ij}\epsilon_{jkm}$

7. Let $\hat{r}(t) \cdot \hat{r}(t) = 1$. Differentiate both sides of the equation. Based on your result, what is the geometrical relationship between $\hat{r}$ and $\dot{\hat{r}}$ (the time derivative of $\hat{r}$)? (This should teach you – or remind you – about a property of polar coordinates. If you have no idea what I'm talking about, don't worry.)

8. The force on a charge q moving with velocity $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ in a magnetic field $\vec{B}$ is $\vec{F} = q(\vec{v} \times \vec{B})$. Since $\vec{\nabla} \cdot \vec{B} = 0$, we can write $\vec{B}$ as $\vec{B} = \vec{\nabla} \times \vec{A}$ where $\vec{A}$ (called the vector potential) is a vector function of x, y, z, t. If the position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ of the charge q is a function of time t show that $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\partial \vec{A}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{A}$. (Hint...you don't necessarily have to use index notation here. Think of how you would define $\mathrm{d}\vec{A}$ in terms of partial derivatives).