Assignment III, PHYS 301 (Classical Mechanics)
Spring 2017
Due 2/10/17 at start of class

1. A little while ago we finished studying air resistance in some detail. Now we're going to use our knowledge of momentum conservation to approach the same basic ideas from a different perspective. Assume you have a long, flat sheet of mass M moving at speed V through a region of space that contains small point-particles of mass m moving at speed v. There are c of these point-particles per unit volume in the space. The sheet is moving in a direction parallel to its surface normal. Assume $M \gg m$ and assume that the point particles do not interact with each other in any way. [Note/hint: in perfectly elastic collisions, the relative velocity between two objects before the collision has to equal the magnitude of the relative velocity between the two objects after the collisions].
   (a) If $v \ll V$, what is the "drag force" induced per unit area on the sheet due to the point particles?
   (b) If $v \gg V$, what is the "drag force" induced per unit area on the sheet due to the point particles? In this case, assume that the component of each particle's velocity in the direction of the sheet's motion is exactly $\pm v/2$. (In reality, these velocities follow a probability distribution function that you'll learn about if you take Thermodynamics. The average speed in each direction, however, is $|\vec{v}|_x = v/2$.)

2. A rod of length L has the following linear mass density:

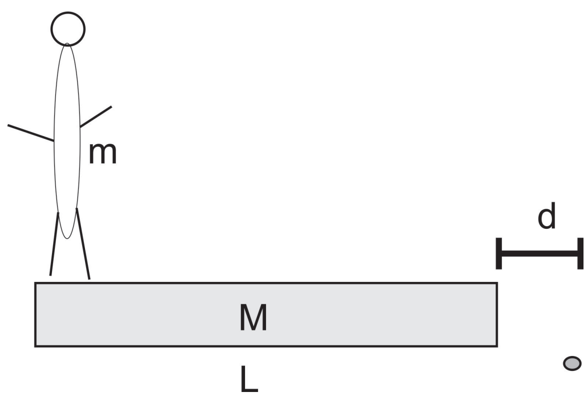
$$\lambda(x) = \begin{cases} 0 & x < 0 \\ \lambda_\circ + kx^5 & 0 \le x \le L \\ 0 & x > L \end{cases}$$

   (Assume $\lambda_\circ$ and k are positive constants).
   (a) What is the mass of the rod?
   (b) Where is the center of mass of the rod?
   (c) What is the moment of inertia of the rod with respect to the z axis? [Note – it is NOT just $(1/3)mL^2$ with m as found in part a]. Note: Leave your answer in terms of $\lambda_\circ$, k, and L only.

3. A man is standing on the very edge of a raft. The raft has a uniform density and has total mass M. The man has mass m and can safely reach a distance past the end of the raft of αd ($\alpha > 1$) past his own center of mass before falling off into the shark-infested waters below. A distance d beyond the other edge of the raft lies a mysterious floating bottle with provisions inside.

   (a) What is the maximum mass m that the man can have and still retrieve the bottle of provisions? (Assume the raft/water interface is frictionless – thus as the man walks, the raft also moves with respect to the water).

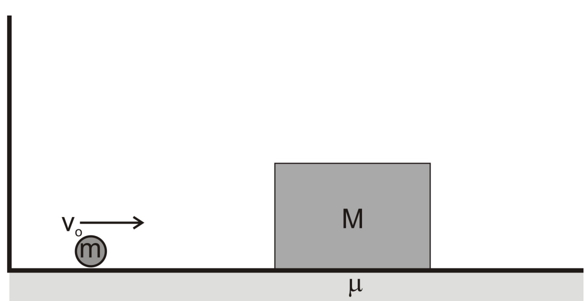
   (b) Assuming the man has mass m you calculated in part (a), where is the center of mass of this system? Use an origin where the man starts at $x = 0$ and the other side of the raft is initially at $x = L$. Your answer should not have an m or and M in it.



4. A ball of mass m and initial speed (to the right) $v_\circ$ bounces back and forth between a fixed wall and a block of mass M with $M \gg m$. The block is initially at rest. Assume that the ball bounces perfectly elastically (both with the wall and with the block). The coefficient of kinetic friction between the block and the ground is μ, whereas there is *no friction between the ball and the ground*. You may assume that $M \gg m$, and you may assume that the distance between the wall and the block is large enough so that the block has time to come to rest between collisions with the ball.

   (a) Show that the ball's speed after n bounces off the block is approximately $\left(1 - \frac{2m}{M}\right)^n v_\circ$.

   (b) How far does the block eventually move from its position at the beginning of the problem?

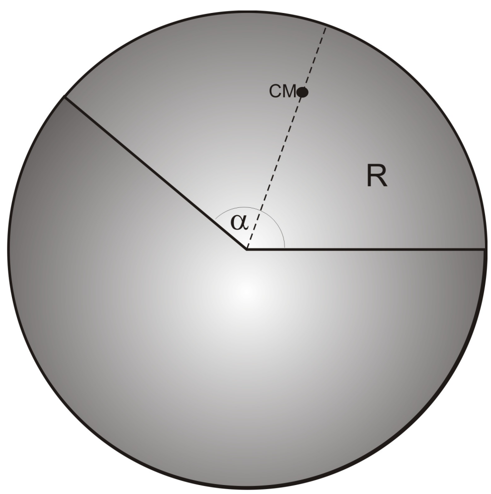


5. A circular sheet of metal is made with that has surface mass density:

$$\sigma = \begin{cases} (ks + \sigma_\circ) & s \leq R \\ 0 & \text{otherwise} \end{cases}$$

with R the radius of the circular sheet, and $\sigma_\circ$ and k some (positive) constants. s is the distance from the origin. From this circular sheet of metal, a wedge is cut out that has central angle α as shown (where $\alpha = \pi$ would indicate that the circle is cut in half, $\alpha = \pi/2$ indicating only a quarter of the circle is retained, etc.; α is constrained between 0 and 2π for obvious reasons). Clearly, the center of mass of the remaining wedge occurs somewhere along the line $\phi = \alpha/2$. Your task is to find how far from the center of the original circle is the center of mass of the wedge cut-out wedge. Leave your answer in terms of k, R, $\sigma_\circ$, and α.

(Hints: If $\alpha = 2\pi$, your answer should be zero, since the center of mass of the whole sheet is at its center. Also, if $\alpha = \pi$ and $k = 0$, your answer should give you $\frac{4R}{3\pi}$ since this is the distance of the center of mass from the center of a uniform semi-circle. Finally, make sure your answer has units of a length!)



6. Let a projectile of mass m be launched on Earth and have the equation of motion $\vec{r}(t) = (v_\circ t\cos\theta)\hat{x} + 0\hat{y} + (v_\circ t\sin\theta - \frac{1}{2}gt^2)\hat{z}$. (For this problem, we're launching a projectile from the origin and neglecting air resistance.)

   (a) Using the implied coordinate system, calculate $\vec{\ell}(t)$ for the particle.

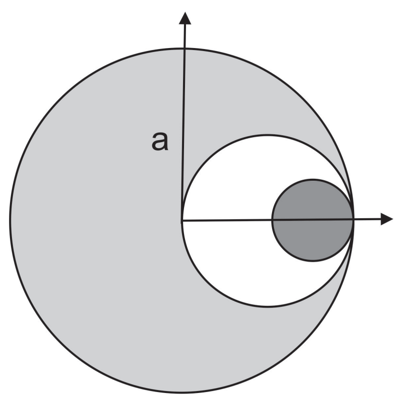
   (b) Take the time derivative above to find the torque on the particle. (Remember, torque is a vector!)

   (c) Find the torque on the particle by calculating $\vec{r} \times \vec{f}$. (Hint – your answer here should match your answer to part (b) if Physics works and stuff).

7. A thin circular lamina of radius a has a smaller circle of radius $a/2$ cut from it in the manner shown (so that the edge of the smaller circle is tangent to both the center of the lamina and the edge of the lamina). Inside the cut region, another circular lamina of radius $a/4$ is placed (so that the edge of the smaller lamina is tangent to both the center of the initially removed region and the edge of the hole as shown). If the initial (lighter grey) lamina has uniform mass density ρ_1, the hole has (clearly) mass density 0, and the small (darker grey) lamina has uniform mass density ρ_2:

   (a) Find the ratio $\frac{\rho_2}{\rho_1}$ if the center of mass is at the left of the darker grey circle.

   (b) Find the center of mass of the system in general (leaving ρ_2 and ρ_1 as parameters, not necessarily in the ratio calculated in part (a)).



8. A thin uniform circular lamina with diameter D has a smaller circle cut out from it – this time with diameter d. The smaller circle is tangent to the larger circle at the outer edge (similar to another problem earlier in this assignment). This system is designed so that the center of mass is exactly at the point directly across from the tangent point of the two circles (See picture below). Find the ratio D/d.

   Note! – when I solved the problem, I ended up with an equation of the form $\alpha\,(D/d)^3 + \beta\,(D/d)^2 + \gamma\,(D/d) + \delta = 0$ with α, β, γ, and δ constants that I'm not going to give you, so you can't cheat and jump in halfway through the answer. What I WILL tell you is that a valid root for D/d given the equation I found was $D/d = 1$. That is NOT the root you want, but this will enable you to factor the cubic equation down to a quadratic equation which you should be able to solve. Only one value of D/d makes physical sense. That's what you are looking for here. Note – the value you get is a pretty famous number (which is why I think this problem is cool. Then again, I'm a nerd – you may be less enthusiastic than I am).