Assignment IV, PHYS 301 (Classical Mechanics)
Spring 2017
Due 2/24/17 at start of class

1. A particle of mass m is subject to the potential energy function $U(x) = -Ax^3 e^{-\alpha x}$. You may assume A and α are positive real constants, and x is constrained between 0 and $+\infty$.

   a) Find $x_\circ$ (the x coordinate associated with the stable equilibrium)

   b) Find $U(x_\circ)$.

   c) What is the angular frequency of small oscillations about the stable equilibrium?

2. This problem deals with another common potential function called the Lennard-Jones potential that was proposed in 1924 by John Lennard-Jones to try to approximate the interaction between a pair of neutral atoms or molecules. (In class, I referred to this as a "6-12" potential). We will use the following form of the Lennard-Jones potential:

$$U(r) = 4\epsilon \left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} \right]$$

   where σ and ϵ are constants. (Note that σ is a radial distance, and at $r = \sigma$ the potential is zero.)

   a) Find the radial position of minimum potential energy for this potential.

   b) Find the value of the potential energy at this minimum.

   c) What is the angular frequency of small oscillations about this minimum for a classical mass of magnitude m? (Keep simplifying to avoid any roots within roots).

3. When setting up the equations for pendulum motion, ultimately you end up with an equation that looks very similar to the equations of motion for a spring system. In fact, if you use the approximation $\sin\theta = \theta$, the two equations are functionally identical. Let's complicate matters a little bit.

   The small angle approximation of $\sin\theta \approx \theta$ is pretty good for small angles, but a better approximation is $\sin\theta \approx \theta - \frac{\theta^3}{6}$. If you use this relationship instead, you find that the force function takes the general form:

$$\vec{F} = -k\left(x - \frac{x^3}{6a^2}\right)\hat{x}$$

   with k and a positive constants.

   a) Find the potential energy function associated with the above force if $U(x = 0) = U_\circ$.

   b) There are three positions of equilibrium for the above force. Find them.

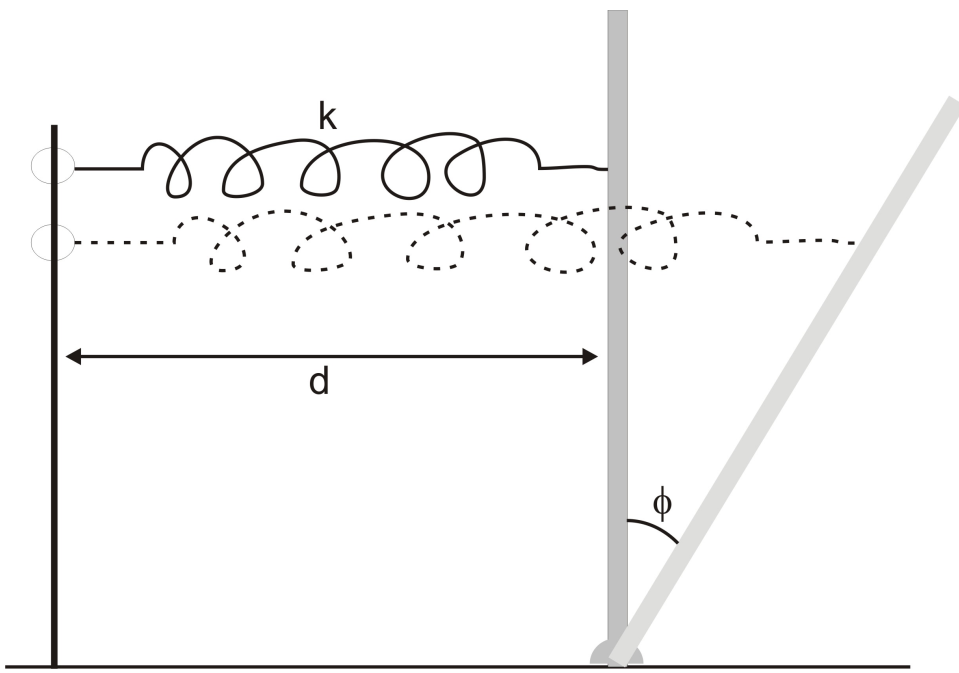
   c) For each of the positions of equilibrium, identify if they are a stable or unstable equilibrium. (Although you might be able to reason this out physically, support your answer with a computation.)

4. A stick of length ℓ and mass m is connected via a spring with spring constant k to a fixed post. The stick is uniform, thus its center of mass is a distance $\ell/2$ from each end. When the stick is oriented vertically, the spring is at its unstretched length of d (equal to the distance between the base of the fixed post and the bottom of the stick. The stick is attached to Earth with a frictionless pivot. The spring is free to move vertically along the post (no friction), but it is affixed to the stick a distance $\alpha\ell$ from the fixed end. ($0 \leq \alpha \leq 1$). Let the angle with respect to the vertical be marked as ϕ. NOTE: The following fact will be useful several times in this problem – you may assume that $mg < 2k\alpha^2\ell$.

   a) Find a potential energy function $U(\phi)$ for this system. Your answer should be in terms only of k, d, m, fundamental constants, and (of course) ϕ. Define $U = 0$ when $\phi = 0$.

   b) Find any point(s) of equilibrium for this system.

   c) For each point of equilibrium, identify the point as a stable or unstable equilibrium.



5. We talked about this in class, so I doubt this comes as a surprise. Consider an undamped simple harmonic oscillator with solution $x(t) = A\cos(\omega_\circ t)$. Calculate the *spatial* average of the kinetic and potential energies in terms of variables A, $\omega_\circ$, k, and/or m. (You may assume that total energy of the system is $\frac{1}{2}kA^2$ and it is conserved at all times and all places). Remember – we found that the *time average* of T was equal to the *time average* of U. In this scenario, you should find them unequal.

6. The squared amplitude of the damped-driven harmonic oscillator can be written:

$$A^2 = \frac{f_\circ^2}{(\omega_\circ^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

   a) Assume β is small compared to $\omega_\circ$. Since the numerator is constant, the expression for $A(\omega)$ is maximized when the denominator is at a minimum. Show that, for $\beta \ll \omega_\circ$, the denominator is minimized when $\omega \approx \omega_\circ\left(1 - \frac{\beta^2}{\omega_\circ^2}\right)$. (Note! I want you to remember that this is not when $\omega = \omega_\circ$!).

   b) Let $\omega_1 = \omega_\circ\left(1 - \frac{\beta^2}{\omega_\circ^2}\right)$. Calculate/find the first non-zero term of $\frac{A(\omega_1)}{A(\omega_\circ)} - 1$.

7. If the amplitude of a damped oscillator decreases to e^{-1} of its initial value after n cycles, show that the frequency of the oscillator must be approximately:

$$\omega \approx \omega_\circ\left(1 - \frac{1}{8\pi^2 n^2}\right)$$

   where $\omega_\circ$ is the frequency of the corresponding oscillator without any damping.