**Homework 4, PHYS 415 (Fluid Mechanics)**
**Spring 2019**
**Due Thursday 31st January 2019 at Beginning of Class**

As always, turn in your legible and annotated work on separate paper.

1. An incompressible fluid of density $\rho$ fills two right circular cylinders of radius $R_1$ and $R_2$ to height $h_1$ and $h_2$, respectively. Find an expression for the work done by the gravitational force in equalizing the levels in the cylinders if a valve in a small tube connecting them at their bases is released, allowing free flow from one cylinder to the other to equalize the fluid heights. You may assume $h_1 > h_2$. Only $h_1$, $h_2$, $R_1$, $R_2$, and fundamental constants may appear in your answer.

2. The velocity components in an unsteady plane flow are given by $v_x = \frac{\alpha t^2}{3x^2}$ and $v_y = 4y\beta t^3 \mathrm{e}^{-\omega t}$ with $\alpha$, $\beta$, and $\omega$ constants.

   a) What are the units of $\alpha$, $\beta$, and $\omega$?

   b) Find the equation of the streamline that passes through the the point $x = x_\circ$ and $y = y_\circ$ at $t = 0$.

   c) Find the parametric equations of the path-line that goes through the point $x = x_\circ$ and $y = y_\circ$ at $t = 0$. (In other words, I want an expressions for $x(t)$ and $y(t)$ with $x(t = 0) = x_\circ$ and $y(t = 0) = y_\circ$.

   d) The answer to the $y$ component in the above part of the problem is messy. Let $\alpha = 300$, $\beta = 1$, and $\omega = 2$ (all with their proper SI units that you should have computed in part (a)). Let a fluid parcel be at $x = 2.45$ m and $y = -1.35$ m at $t = 0$. Where will the fluid parcel be at $t = 0.4$ s? (I'm looking for a numerical answer – give me $x$ and $y$ coordinates to 3 significant figures).

3. Show (either by direct computation or through convincing argument) that all of the elements on the diagonal of the rotation tensor $r_{ij}$ are zero and that $r_{ij}^T = -r_{ij}$ (aka $r_{ij}$ is antisymmetric).

MORE ON BACK

4. In class, I asked if anyone had solved the compressible atmosphere hydrostatic balance problem, and I was surprised at how few of you had seen it (I asked Dr. Williams, and he suggested that maybe that those of you who took Synoptic Meteorology and/or Thermodynamics in the past may not have been as forthcoming as you should have been, because you've definitely seen it). Either way, we'll attack it here. To mimic the Earth's atmosphere, let's assume that you have a tall (many-km) column of gaseous nitrogen within the Earth's gravitational field. The gas is in thermal equilibrium at constant temperature $T$ but the concentration of gas molecules per unit volume $n$ is allowed to vary with height (in other words, we're explicitly *not* talking about an incompressible fluid – which makes sense; you can squeeze gas without too much difficulty). Let's choose $z = 0$ to be the surface of the Earth and let $z$ increase upwards, so that $n(z)$ will decrease with increasing $z$.

   a) Similar to what I did for an incompressible fluid, use force balance on a parcel of the gas to justify $P(z) = P(z + \Delta z) + mg[n(z)](\Delta z)$ where $m$ is the mass of an average Nitrogen molecule.

   b) Use the above equation to construct a differential equation, and solve it to find $P(z)$ given that $P(z = 0) = P_\circ$. Note – you will want to use the ideal gas law $PV = NkT$ to replace $n(z)$ in terms of more convenient quantities relevant to your differential equation. Even though our column of nitrogen is tall, you may assume $g$ is constant.

   c) The height at which the number density falls to 1/e of its surface value is called the "e-folding" height. Use your result to part (b) above to numerically determine the e-folding height of the Earth's atmosphere if it were isothermal at 300K.

   d) The atmosphere of Mars is almost entirely Carbon Dioxide. Look up any requisite quantities necessary for Mars (remember, $g$ is different there!) and numerically determine the e-folding height of the Martian atmosphere.