Assignment V, PHYS 301 (Classical Mechanics)
Spring 2015
Due 2/13/15 at start of class

1. Consider a rocket traveling with initial mass $m_\circ$ and initial velocity $v_\circ$ to the right. At time $t = 0$, the rocket starts to eject rocket fuel to the left at speed v_{ex} (with respect to the rocket). As it burns its fuel, the rocket speeds up, but ejects mass at a constant rate.

    a) Find the mass of the rocket at the time when the rocket has its maximal momentum.

    b) What is the instantaneous velocity of the rocket at the time when the rocket has its maximal momentum?

2. The force acting on a particle of mass m is:

$$\vec{F} = k\left[(2xy + x^2)\hat{x} + (x^2 + y^2)\hat{y} + z^2\hat{z}\right]$$

    with k a positive constant.

    a) Show that $\vec{F}$ is a conservative force.

    b) What is the associated potential energy function for the force $\vec{F}$? Assume that the potential energy at the origin is given by $U_\circ$.

    c) What work is done by the force $\vec{F}$ as a particle goes from the point $\hat{x} + 2\hat{y} + 3\hat{z}$ to $-3\hat{x} + 4\hat{y} - 5\hat{z}$?

    d) Suppose the particle (of mass m) passes through the origin with a speed $v_\circ$ directed in such a fashion that it ultimately passes through the points $\hat{x} + 2\hat{y} + 3\hat{z}$. What is the speed of the particle when it passes through the point $\hat{x} + 2\hat{y} + 3\hat{z}$?

3. When setting up the equations for pendulum motion, ultimately you end up with an equation that looks very similar to the equations of motion for a spring system. In fact, if you use the approximation $\sin\theta = \theta$, the two equations are functionally identical. Let's complicate matters a little bit.

    The small angle approximation of $\sin\theta \approx \theta$ is pretty good for small angles, but a better approximation is $\sin\theta \approx \theta - \frac{\theta^3}{6}$. If you use this relationship instead, you find that the force function takes the general form:

$$\vec{F} = -k\left(x - \frac{x^3}{6a^2}\right)\hat{x}$$

    with k and a positive constants.

    a) Find the potential energy function associated with the above force if $U(x = 0) = U_\circ$.

    b) There are three positions of equilibrium for the above force. Find them.

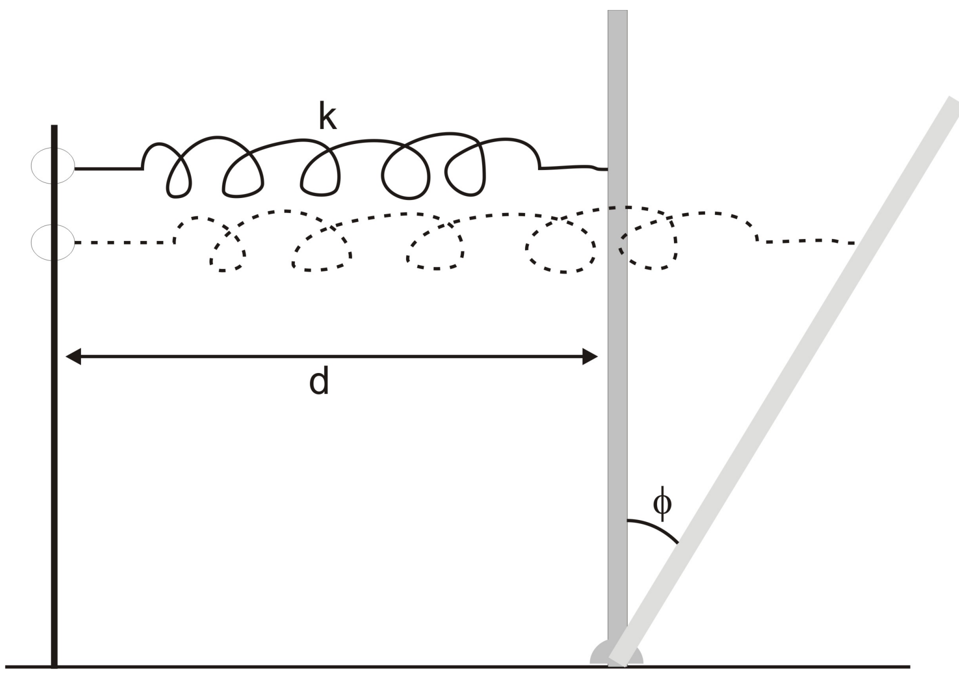
    c) For each of the positions of equilibrium, identify if they are a stable or unstable equilibrium. (Although you might be able to reason this out physically, support your answer with a computation.)

4. A stick of length ℓ and mass m is connected via a spring with spring constant k to a fixed post. The stick is uniform, thus its center of mass is a distance $\ell/2$ from each end. When the stick is oriented vertically, the spring is at its unstretched length of d (equal to the distance between the base of the fixed post and the bottom of the stick. The stick is attached to Earth with a frictionless pivot. The spring is free to move vertically along the post (no friction), but it is affixed to the stick a distance $\alpha\ell$ from the fixed end. ($0 \leq \alpha \leq 1$). Let the angle with respect to the vertical be marked as ϕ. NOTE: The following fact will be useful several times in this problem – you may assume that $mg < 2k\alpha^2\ell$.

   a) Find a potential energy function $U(\phi)$ for this system. Your answer should be in terms only of k, d, m, fundamental constants, and (of course) ϕ. Define $U = 0$ when $\phi = 0$.

   b) Find any point(s) of equilibrium for this system.

   c) For each point of equilibrium, identify the point as a stable or unstable equilibrium.



5. A solid triangular wedge is placed on Earth. The hypotenuse (of length D) has a surface that is frictionless. To the top of the wedge is affixed a (massless) spring with spring constant k and equilibrium length $x_\circ$. To the bottom of the wedge, a massless spring with spring constant $3k$ and equilibrium length $x_\circ$ is placed. A point mass m is attached to both springs. $D = 2x_\circ$ so that, if this system were placed in space (in the absence of gravity), the system would be in a stable equilibrium with the point mass exactly halfway down the hypotenuse. The angle of inclination of this wedge is ϕ as shown. Let the variable ℓ indicate the distance of the mass from the bottom right corner of the triangle.

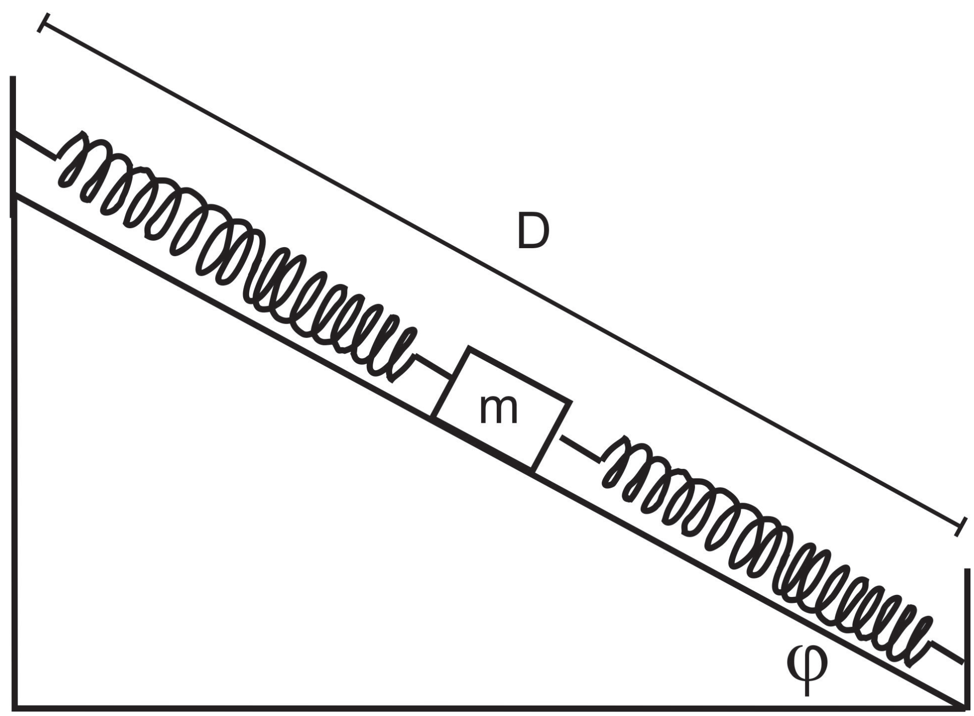
   a) Letting $\ell = x_\circ$ correspond to the case where $U(\ell) = 0$, write down a potential energy function for this system (e.g. write down $U(\ell)$. (Make sure that you get $U(\ell) = 0$ when $\ell = x_\circ$!

   b) Find the net force on the block as a function of ℓ via $\vec{F} = -\vec{\nabla}U$.

   c) Find the equilibrium position of the block as a function of $x_\circ$, ϕ, k, m, and g only. (Hint / something for you to check – what should your equilibrium ℓ be when $\phi = 0$?)

   d) Is your equilibrium calculated in part (c) a stable equilibrium? Justify your answer with computations.

   e) If the block is pulled to the position $\ell = \frac{5x_\circ}{3}$ and released, what is the block's speed as it moves through the point $\ell = x_\circ$? (Leave your answer in terms of $x_\circ, \phi, k, m$, and g only.)



(MORE ON BACK!)

6. Consider the setup shown below. Both systems are in static equilibrium (no motion) and the same mass M is applied to each. In fact, the right-hand side graphic is basically the same system as shown on the left-hand side, except with the tiny string between the strings cut and after a new equilibrium has been reached. All springs are massless with spring constant k and equilibrium length $x_\circ$, and there is gravity in this problem. On the left, the spare strings drawn are not supplying any tension, though they are "almost" under tension. (If either string was extended any further, the "spare" strings would be taught. The left string is the same length as the stretched upper spring on the left. The right string is the same length as the stretched lower spring on the left.)

   You may not need this reminder, but recall that when you have a spring attached to another spring, the applied force on both springs are the same.

   a) What is the distance between the ceiling and the mass on the left hand side of the graphic below?

   b) The right hand picture shows what would happen if the tiny bit of string connecting the bottom of the upper spring and the top of the bottom spring were to be cut. Calculate the new distance between the ceiling and the mass.

   c) Your answer to parts (a) and (b) should clearly both be larger than $2x_\circ$ (the combined unstretched length of both springs). Let your answer to part (a) be A and your answer to part (b) be B. Calculate the ratio $\frac{A-2x_\circ}{B-2x_\circ}$ and comment on your answer.