Assignment VI, PHYS 412 (MAP)
Fall 2014
Due 10/03/14 at start of class

1. Write the following equations in index notation. The tilde symbol above a variable (e.g. $\widetilde{A}$) indicates a matrix. (All of the equations below should be well-defined).

   a) $\vec{f} = \vec{a} + \vec{b} \times \vec{c}$
   b) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{f} \cdot \vec{g})|\vec{h}|^2\vec{k}$
   c) $\widetilde{A}\vec{x} - (\vec{y} \times \vec{z}) = \vec{b}$
   d) $\vec{\nabla}\phi + \vec{\nabla} \times \vec{a} = (\vec{\nabla} \cdot \vec{b})\vec{c}$
   e) $(\nabla^2\phi)\vec{a} + \widetilde{B}\vec{c} = \vec{d} \times \vec{f}$
   f) $\frac{\partial^2 \vec{u}}{\partial t^2} = |\vec{v}|^2\nabla^2\vec{u}$
   g) $\frac{-\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$
   h) $\vec{u} + (\vec{a} \cdot \vec{b})\vec{v} = |\vec{a}|^2(\vec{b} \cdot \vec{v})\vec{a}$

2. Use index notation to demonstrate the following:

   a) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$
   b) $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$
   c) $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$
   d) $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2\vec{a}$

3. The Helmholtz theorem tells us that any vector $\vec{F}$ can be written as the negative gradient of a scalar field plus the curl of a vector field. i.e.:

$$\vec{F} = -\vec{\nabla}V + \vec{\nabla} \times \vec{E}$$

   a) Rewrite the above vector equation in index notation.
   b) Manipulate the expression in part (a) to show that $\vec{\nabla} \times \vec{F} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E}$. (This doesn't mean just referencing some source that argues that this is true; it means manipulating the index notation to clearly progress from the expression in part (a) to something equivalent to the above in index notation).

MORE ON BACK!!!

4. Use index notation to simplify / rewrite the following to the number of terms specified. (Let ϕ be an arbitrary scalar function and $\vec{a}$ be an arbitrary vector function):

   a) $\vec{\nabla} \times (\phi \vec{\nabla} \phi)$ (when simplified, this can be written as a single term).

   b) $\vec{\nabla} \cdot (\phi \vec{\nabla} \phi)$ (when rewritten, this should have two terms).

   c) $\vec{\nabla} \times (\phi \vec{a})$ (when rewritten, this should have two terms).

5. Simplify the following expressions:

   a) $\delta_{ij}\delta_{ij}$

   b) $\delta_{ij}\delta_{jk}\delta_{ki}$

   c) $\epsilon_{ijk}\epsilon_{mjk}$

   d) $\epsilon_{ijk}\epsilon_{ijk}$

   e) $\delta_{ij}\epsilon_{jkm}$

6. Let $\hat{r}(t)\cdot\hat{r}(t) = 1$. Differentiate both sides of the equation. Based on your result, what is the geometrical relationship between $\hat{r}$ and $\dot{\hat{r}}$ (the time derivative of $\hat{r}$)? (This should teach you – or remind you – about a property of polar coordinates. If you have no idea what I'm talking about, don't worry.)

7. The force on a charge q moving with velocity $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ in a magnetic field $\vec{B}$ is $\vec{F} = q(\vec{v} \times \vec{B})$. Since $\vec{\nabla} \cdot \vec{B} = 0$, we can write $\vec{B}$ as $\vec{B} = \vec{\nabla} \times \vec{A}$ where $\vec{A}$ (called the vector potential) is a vector function of x, y, z, t. If the position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ of the charge q is a function of time t show that $\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = \frac{\partial \vec{A}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{A}$. (Hint...you don't necessarily have to use index notation here. Think of how you would define $\mathrm{d}\vec{A}$ in terms of partial derivatives).