Assignment VI, PHYS 301 (Classical Mechanics)
Spring 2014
Due 2/28/14 at start of class

Due to all of the class cancelations and the test fiasco, we've gotten off of our planned schedule a little bit. Consequently, I'm combining two homework assignments into one – and giving you a bit longer than usual to do it (almost 2 full weeks). I WILL BE COUNTING THIS HOMEWORK DOUBLE! Everything above the horizontal line gets one homework grade, everything below the horizontal line gets a different, second homework grade. Please plan accordingly.

1. Consider a vertically oriented pendulum of length ℓ with a bon of mass m at its end. The whole pendulum is submersed in a viscous fluid. The bob undergoes small oscillations, but the fluid retards the bob's motion with a resistive force proportional to the speed with $|\vec{F}_{\text{drag}}| = 2m\sqrt{\frac{g}{\ell}}\left(\ell\dot{\theta}\right)$. The bob is initially pulled back at $t = 0$ with $\theta = \theta_\circ$ and $\dot{\theta}(t = 0) = 0$. Find the angular displacement θ and angular velocity $\dot{\theta}$ as a function of time.

2. Consider a simple harmonic oscillator with solution $x(t) = A\cos(\omega_\circ t)$. Calculate the *time* average of the kinetic and potential energies over one cycle, and show that these quantities are equal.

3. Consider a simple harmonic oscillator with solution $x(t) = A\cos(\omega_\circ t)$. Calculate the *space* average of the kinetic and potential energies. (You may assume total energy is conserved; that might come in handy). Discuss the results – especially in light of your answer to the previous problem.

4. The amplitude of a damped-driven harmonic oscillator can be written as:

$$A^2 = \frac{f_\circ^2}{(\omega_\circ^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

   a) Assume β is small compared to $\omega_\circ$. Since the numerator is constant, the expression for $A(\omega)$ is maximized when the denominator is at a minimum. Show that, for $\beta \ll \omega_\circ$, the denominator is minimized when $\omega \approx \omega_\circ\left(1 - \frac{\beta^2}{\omega_\circ^2}\right)$. (Note that this is slightly less than $\omega_\circ$).

   b) Let $\omega = \omega_\circ\left(1 - \frac{\beta^2}{\omega_\circ^2}\right)$. Plug this back in for ω in the above expression and find an approximate expression for $A(\omega)$ at its maximum.

5. If the amplitude of a damped oscillator decreases to e^{-1} of its initial value after n periods, show that the frequency of the oscillator must be approximately:

$$\omega = \omega_\circ \left(1 - \frac{1}{8\pi^2 n^2}\right)$$

where $\omega_\circ$ is the frequency of the corresponding oscillator without any damping.

6. See page 185 of your text. You are tasked to repeat the calculations given in example 5.3 but with the following parameters instead:

$$\omega = 2\pi \qquad \omega_\circ = \omega/3 \qquad \beta = 0.4\omega_\circ \qquad f_\circ = 1000$$

and with the initial conditions $x_\circ = 0$ and $v_\circ = 0$. Plot $x(t)$ for $0 \le t \le 8$ and compare with the plot of example 5.3.

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7. Find the x coordinate of the point on the graph of $y = \sqrt{x}$ that is nearest to the point $3\hat{x} + 0\hat{y}$.

8. An isosceles triangle has two side lengths ℓ and ℓ.

   a) Let the angle between the edges of length ℓ and ℓ be ϕ. Find ϕ that maximizes the area of the triangle.

   b) What is the area of the triangle when the area is maximized (in terms of ℓ)?

9. Let $x = x(t)$. Let the integral S be defined via:

$$S = \int_0^1 \left(t^2 + x^2 + (x')^2\right) \mathrm{d}t$$

subject to the boundary conditions $x(t = 0) = 0$ and $x(t = 1) = 2$.

   a) Find the functional form of x that minimizes this integral.

   b) Numerically (using a calculator or a computer algebra system) evaluate the value of the integral when using this functional form of x.

   c) Numerically (by hand) calculate the value of S if $x(t) = 2t$ (which also obeys the boundary conditions). (Hint – if the calculus of variations works, you should get a larger answer for part (c) than for part (b)). (They are pretty close, though).

10. The corners of a rectangle lie on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

    a) Where should the corners be located in order to maximize the area of the rectangle?

    b) What fraction of the area of the ellipse is covered by the rectangle with maximum area?

11. Find and describe the path $x = x(t)$ for which the integral $\int_{t_1}^{t_2} \left(1 + \frac{1}{t^3}(x')^2\right) \mathrm{d}t$ is stationary.

12. A body is released from a height of 20 meters (in a vacuum) and approximately 2 seconds later it strikes the ground. The equation for the distance of fall z during time t could conceivably have any of the following forms (where g has different units in the three expressions):

$$z = gt \qquad z = \frac{1}{2}gt^2 \qquad z = \frac{1}{4}gt^3$$

    all of these yield $z = 20$ meters for $t = 2$ seconds (for the sake of mathematical simplicity, we will approximate $g = 10$ – in appropriate (but different) units – for all three systems). Show (via direct computation for the three cases) that the correct form gives the smallest value for the integral of the Lagrangian.