Assignment VI, PHYS 301 (Classical Mechanics)
Spring 2017
Due 3/17/17 at start of class

1. A circular cone holds volume $V = \frac{1}{3}\pi r^2 h$ and has a total surface area $A = \pi r^2 + \pi r(r^2 + h^2)^{1/2}$ with h the height of the cone and r the radius of the cone at its base.

   a) What is the radius of the cone of volume V that minimizes the total surface area A? (Leave your answer in terms of V only).

   b) Find the cone angle for the cone of volume V that minimizes the total surface area A. (The cone angle is the angle from one wall inside the cone to the other. If the cone was nearly a flat plane, the cone angle would be approaching $\pi/2$ radians). (You may use a calculator for this part; you may give your answer in degrees or radians, just be clear which).

2. Let $x = x(t)$ and let the integral S be defined via:

$$S = \int_0^1 (t^2 + x^2 + \dot{x}^2)\mathrm{d}t$$

   subject to the boundary conditions $x(t = 0) = 0$ and $x(t = 1) = 2$.

   a) Find the functional form of x that minimizes this integral.

   b) Numerically (using a calculator or a computer algebra system) evaluate the value of the integral when using this functional form of x.

   c) Numerically (by hand) calculate the value of S if $x(t) = 2t$ (which also obeys the boundary conditions). (Hint – if the calculus of variations works, you should get a larger answer for part (c) than for part (b)). [They are pretty close, though.]

3. Examine Figure 7.6 in your text; what is depicted is an "Atwood Machine" (you may remember these from PHYS 111). We will use the same setup described in the figure; two masses m_1 and m_2 are suspended by a massless inextensible string that passes over a pulley of radius R. We will use the variable x to describe the state of the system. Unlike the figure caption, however, we will *not* assume the pulley is massless. Though the pulley's axis is fixed (it isn't moving vertically), we will now state that the pulley has moment of inertia I. The string stays in contact with the pulley without slipping as it rotates.

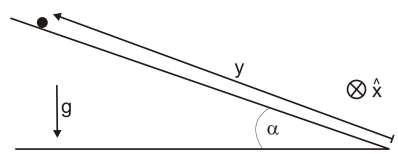
   a) Write the new Lagrangian for this system (taking into account the kinetic energy associated with the pulley). As is always the case, we need you to write the entire Lagrangian in terms of x and known parameters (e.g. m_1, m_2, I) only.

   b) Use the Euler-Lagrange relations to find an expression for the acceleration $\ddot{x}$.

   c) Is this acceleration larger or smaller than the case solved in the text for a massless pulley?

4. Let a point mass m move on a semi-infinite frictionless surface tilted at angle α with respect to the horizontal. Gravity points directly down. Let the variable y correspond to the distance from the vertex of the frictionless wedge as shown, and let $\hat{x}$ point into the page. (You may choose the origin of the x axis wherever you want). This is a system with two degrees of freedom; once you've chosen the origin of the x-axis, you can uniquely describe the position of the mass with two coordinates – x and y. (See picture below to try and clear up any ambiguity).

   a) Write the Lagrangian of this system in terms of variables x and y.
   b) Find the differential equations governing $x(t)$ and $y(t)$.
   c) Solve the differential equation for $x(t)$ (you may have a couple of undetermined constants).
   d) Solve the differential equation for $y(t)$ (again, you may have a couple of undetermined constants).
   e) If $x(t=0) = x_\circ$, $\frac{\mathrm{d}x}{\mathrm{d}t}\big|_{t=0} = v_{x\circ}$, $y(t=0) = y_\circ$, and $\frac{\mathrm{d}y}{\mathrm{d}t}\big|_{t=0} = v_{y\circ}$, write expressions (without arbitrary constants) for $x(t)$ and $y(t)$.



5. Look back at the inclined plane shown in the previous problem. This problem uses a similar (though not identical) geometry. In this system, we will be changing the inclined plane over time, so that $\alpha = \omega t$ for $t \geq 0$ with ω some positive constant. (The incline gets steeper over time).

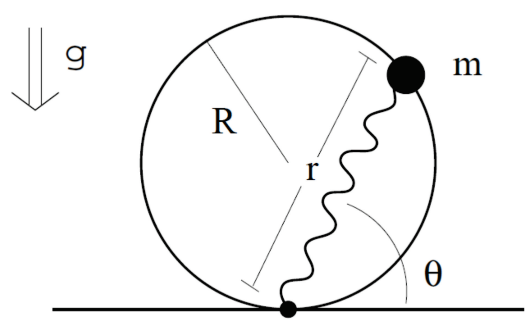
   a) Neglect the x direction entirely and treat this as a one-dimensional system with y as your generalized coordinate. Show that $\ddot{y} - \omega^2 y = -g \sin \omega t$.
   b) Solve the above differential equation assuming that $y(t=0) = y_\circ$ and the initial velocity is 0.

6. A bead of mass m slides without friction on a fixed vertical hoop of radius R. The bead moves under the combined action of gravity and a spring attached to the bottom of the hoop. *For simplicity, we assume the equilibrium length of the spring is zero*, so the force due to the spring is $-kr$ where r is the instantaneous length of the spring. Assume that the bead can pass unhindered through the point at the bottom of the loop.

   a) Write down the Lagrangian for the problem, using θ as drawn as the generalized coordinate. (Note; that means you'll have to be clever and find r as a function of θ; since R is a constant in the problem it is allowed to show up in your final expression. r may not.)

   b) Use the Euler-Lagrange equation to come up with the general differential equation of motion for the problem (in terms of θ and its time derivatives). Simplify so that the $\ddot{\theta}$ term has a coefficient of 1.

   c) Show that the angular frequency of small oscillations about the bottom of the loop is $\left(\frac{g}{R} + \frac{k}{m}\right)^{1/2}$.



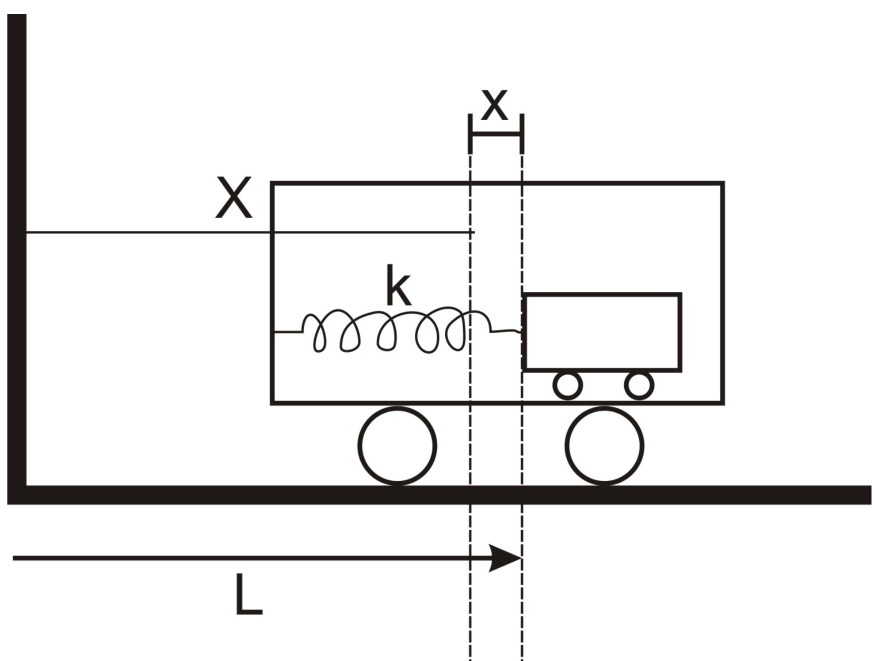
7. Examine the figure below. A cart of mass m is mounted inside a larger cart of irrelevant mass. The two carts are attached via a spring (with spring constant k) as shown. Let the distance the small cart is displaced *with respect to the large cart* be x. (In other words, when $x = 0$, the spring is neither compressed nor extended). Let X be the distance from a fixed point (say a wall) to the point where $x = 0$. Thus, the position of the small cart with respect to the same wall would be $L = X + x$.

   The large cart is exposed to a forced oscillation in such a way that $X = A\cos(\omega t)$ with ω and A constant. (Don't worry about the mechanism forcing this oscillation – if you want, just envision someone applying a sinusoidal motion to the big cart by hand).

   a) Set up the Lagrangian and use it to find a differential equation for x. Your answer should only have A, ω, k, m, t, and (of course) x and its derivatives involved.

   b) The differential equation you found in part (a) should look really familiar. Use the particular solution of that differential equation to describe (in words) the behavior of $x(t)$ if $\omega \ll \sqrt{\frac{k}{m}}$. (Comment on the amplitude and phase of $x(t)$ compared to $X(t)$).

   c) Do the same as part (b) except describe the situation if $\omega \gg \sqrt{\frac{k}{m}}$



8. A rigid pendulum of length L has a mass m on its end and another mass m somewhere on the support between the pivot and the end at a distance ℓ from the pivot. Use ϕ (the angle from the vertical) as your generalized coordinate. Note that ℓ is *not* a generalized coordinate; the position of the mass on the shaft is fixed at some constant $\ell < L$.

   You may assume that the only masses in the system are the two point masses of mass m.

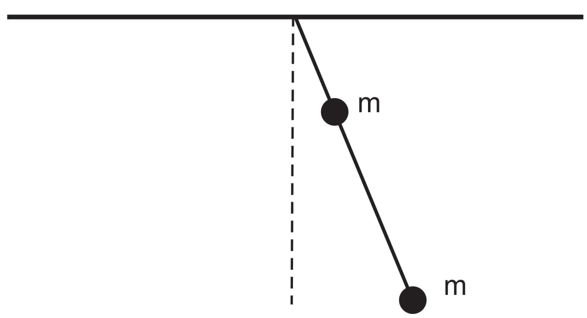
   a) Write the Lagrangian for this system.

   b) Use the Euler-Lagrange Equations to find a homogeneous differential equation for $\ddot{\phi}$.

   c) For small angles, find the angular frequency of small oscillations.

   d) Although ℓ is fixed, let's say that you are able to change it. (Imagine, for example, that the mass at ℓ has a lot of static friction with the shaft so that it doesn't move while in oscillation, but can still be moved if you physically take the pendulum down and slide it along the shaft, then replace the system with ℓ taking on a new value.) Find the value of ℓ between 0 and L that maximizes or minimizes this frequency of small oscillations.

   e) Does positioning the mass on the shaft at the position ℓ found in part (d) maximize or minimize the frequency of small oscillations? Clearly explain how you know.



9. Below is a picture of block of mass m_2 on an inclined plane of mass m_1. There is no friction between the block and the plane, and *there is also no friction between the plane and the floor*. This system is on the surface of the Earth, so gravity is a factor. Use generalized coordinate x_1 to indicate the position of the inclined plane and generalized coordinate x_2 to indicate the distance between the top of the inclined plane and the block. The inclined plane has inclination angle β as shown. You may assume that, for all times of interest, the block remains on top of the plane. (Don't worry about what happens after the block leaves the surface of the plane).

   a) Find the Lagrangian for this system.

   b) Use your Lagrangian to develop two (coupled) differential equations.

   c) Although these differential equations are coupled, they aren't that hard to uncouple. Solve them. (Hint: solve one of them for $\ddot{x}_1$ and substitute this into the other differential equation). You should be able to get $\ddot{x}_1 = c_1$ and $\ddot{x}_2 = c_2$ with c_1 and c_2 each functions of m_1, m_2, g, and β. Remember, fractions within fractions are evil. Your answer also should make sense if $\beta = 0$ or $\beta = \frac{\pi}{2}$. (If you'd like, you could also check your answers with a Newtonian analysis – though doing this with Newtonian mechanics is trickier than it seems!)