**Assignment VII, PHYS 272 (MAP)**
**Fall 2014**
**Due 10/24/14 at start of class**

1. Let $\vec{H} = (2xy - z^3)\hat{x} + x^2\hat{y} + (\alpha xz^2 - 1)\hat{z}$.

   a) If $\vec{H}$ is a conservative vector field, what must $\alpha$ be?

   b) Using the value of $\alpha$ found in part (a), calculate:

   $$\int_{\vec{P}_1}^{\vec{P}_2} \vec{H} \cdot \mathrm{d}\vec{\ell}$$

   if $\vec{P}_1 = \vec{0}$ and $\vec{P}_2 = \hat{x} + 2\hat{y} + 3\hat{z}$. (Note; if you've done part (a) right, this shouldn't depend on your path. A good check on part (a) would be to do this via two different routes and see if you come up with the same thing).

   c) Using the value of $\alpha$ found in part (a), find $\phi$ if $\vec{H} = \vec{\nabla}\phi$.

2. Let us define:

   $$\vec{a} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (3r\sin\theta\cos\phi)\hat{\phi}$$
   $$\vec{b} = s(3 + \sin^2\phi)\hat{s} + s\sin\phi\cos\phi\hat{\phi} + 3z^2\hat{z}$$

   (If it wasn't obvious from context or the variables used, $\vec{a}$ is described in spherical coordinates, and $\vec{b}$ is described in cylindrical coordinates). Find the following:

   a) The divergence of $\vec{a}$.

   b) The curl of $\vec{a}$.

   c) The divergence of $\vec{b}$.

   d) The curl of $\vec{b}$.

3. Calculate $\oiint \vec{L} \cdot \mathrm{d}\vec{a}$ for $\vec{L} = x\cos^2 y\hat{x} + xz\hat{y} + z\sin^2 y\hat{z}$ over the surface of a sphere with center at the origin and radius 3.

4. Given $\varphi = xy + yz + z\sin x$, find:

   a) $\vec{\nabla}\varphi$

   b) Evaluate $\vec{\nabla}\varphi$ at $-3\hat{y} + \hat{z}$.

5. Calculate $\oint \vec{M} \cdot \mathrm{d}\vec{\ell}$ around the boundary of the square with vertices $\langle 1,0\rangle$, $\langle 0,1\rangle$, $\langle -1,0\rangle$, $\langle 0,-1\rangle$ if $\vec{M} = x^2\hat{x} + 3xy\hat{y}$.

6. What is wrong with the following "proof" that magnetic fields do not exist?

   By electromagnetic theory, $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. (This part is totally legitimate; the error in the proof is not based on the previous sentence). Using the divergence theorem, we can write:

$$\iiint \left(\vec{\nabla} \cdot \vec{B}\right) \mathrm{d}V = 0 = \oiint \vec{B} \cdot \mathrm{d}\vec{a}$$

   However, if we then substitute in $\vec{B} = \vec{\nabla} \times \vec{A}$ we can use Stokes' theorem to write:

$$\iint (\vec{\nabla} \times \vec{A}) \mathrm{d}\vec{a} = \oint \vec{A} \cdot \mathrm{d}\vec{r}$$

   Since $\oint \vec{A} \cdot \mathrm{d}\vec{r} = 0$, we know that $\vec{A}$ is conservative, or $\vec{A} = \vec{\nabla}\phi$ for some scalar function $\phi$. Then we have $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times (\vec{\nabla}\phi) = 0$, implying $\vec{B} = 0$. (Note – this "proof" *is* wrong. Magnetic fields do exist! The error is somewhere in the Mathematical argument above; detailed knowledge of E&M is not necessary.)

7. Let $x$ be a function of $t$.

   a) If the acceleration $a = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} \equiv \ddot{x} = A\sin(\omega t)$ for some constants $A$ and $\omega$, what is the solution for $x(t)$?

   b) The above solution should have two arbitrary constants besides $A$ and $\omega$. If $\dot{x}$ at $t = 0$ is 0 and $x$ at $t = 0$ is -3, what is the solution for $x(t)$?

8. The differential equation for a particle moving vertically under the presence of air-resistance can be written as:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - \beta v$$

   With $\beta$ some constant associated with the physical properties of the system.

   a) What is the steady-state value (e.g. free-fall speed) of this system? (Hint...what does steady-state mean in terms of $v$?)

   b) If a particle of mass $m$ is dropped from rest, find $v(t)$. (In other words, solve this differential equation under the initial condition $v(t = 0) = 0$). A change of variable may be helpful.

9. A spherical drop of an unknown airborne liquid is known to grow at a rate proportional to its surface area, e.g.:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \alpha(4\pi r^2)$$

with $m$ the mass of the drop and the proportionality constant $\alpha$ unknown. The mass of the drop is equal to its volume multiplied by a constant (known) mass density $\rho_\circ$. If the spherical drop originally has a radius of 20 $\mu$m and evaporates to a radius of 10 $\mu$m after 7 minutes, how long will/does it take for the radius of the initially 20 $\mu$m drop to evaporate down to a sphere with a 2 $\mu$m radius?

10. A spherical ball at the origin of radius $R$ has, enclosed within it, a mass density that follows the relationship $\rho(r, \theta) = 3\alpha r^3 \cos^2(\theta)$. (Obviously, this relationship is only valid for $r < R$, since – for $r > R$ – we're not talking about the "innards" of the ball anymore). For a hypothetical ball of radius $r$ centered at the origin, what would be the mass enclosed? ($r$ could, in principle, be smaller than or larger than $R$; your answer should be a piecewise defined function of the type:

$$m(r) = \begin{cases} m_1(r) & r < R \\ m_2(r) & r > R \end{cases}$$

Your task here is to find $m_1(r)$ and $m_2(r)$. (Note, $\mathrm{d}V$ in spherical coordinates is NOT $\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$, but rather $r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$). Note also, don't read anything into this – this doesn't totally fit into this section of the class, but it doesn't really fit anywhere else, either – but people have always struggled with doing this sort of thing in E&M and I want you to have an advantage when you get to that class.