Assignment VII, PHYS 301 (Classical Mechanics)
Spring 2015
Due 3/13/15 at start of class

1. Let $x = x(t)$. Let the integral S be defined via:

$$S = \int_0^1 \left(t^2 + x^2 + (\dot{x})^2\right) \mathrm{d}t$$

subject to the boundary conditions $x(t = 0) = 0$ and $x(t = 1) = 2$.

   a) Find the functional form of x that minimizes this integral.

   b) Numerically (using a calculator or a computer algebra system) evaluate the value of the integral when using this functional form of x.

   c) Numerically (by hand) calculate the value of S if $x(t) = 2t$ (which also obeys the boundary conditions). (Hint – if the calculus of variations works, you should get a larger answer for part (c) than for part (b)). (They are pretty close, though).

2. Find and describe the path $x = x(t)$ for which the integral $\int_{t_1}^{t_2} \left(1 + \frac{1}{t^3}(\dot{x})^2\right) \mathrm{d}t$ is stationary.

3. Start with the solution to the brachistrochrone problem and show that the time required for a particle to move (frictionlessly) to the minimum point of the cycloid is $\pi\sqrt{\frac{a}{g}}$ *no matter what starting point is chosen!* Here's a particularly good starting point:

$$x(\theta) = a(\theta - \sin\theta)$$
$$y(\theta) = a(1 - \cos\theta)$$

remember, in the book's coordinate system, y is positive in the downward direction, meaning the minimum point occurs when $\theta = \pi$. This integral should come in extremely helpful:

$$\int_{\theta_\circ}^{\pi} \frac{\mathrm{d}\theta\sqrt{(1 - \cos\theta)}}{\sqrt{(\cos\theta_\circ - \cos\theta)}} = \pi$$

(You can just use that as a result without showing that it is true in your answer).

4. A body is released from a height of 20 meters (in a vacuum) and approximately 2 seconds later it strikes the ground. The equation for the distance of fall z during time t could conceivably have any of the following forms (where g has different units in the three expressions):

$$z = gt \qquad z = \frac{1}{2}gt^2 \qquad z = \frac{1}{4}gt^3$$

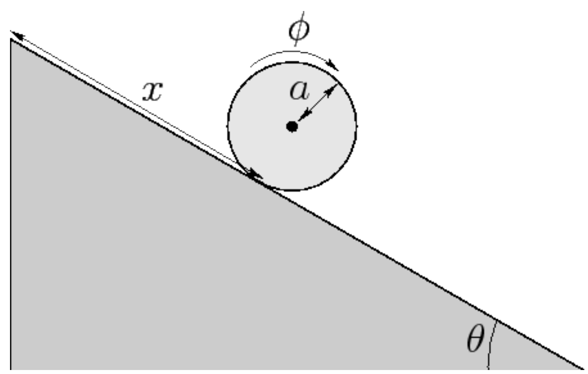
all of these yield $z = 20$ meters for $t = 2$ seconds (for the sake of mathematical simplicity, we will approximate $g = 10$ – in appropriate (but different) units – for all three systems). The kinetic energy is going to be equal to $\frac{1}{2}m\dot{z}^2$ and the potential energy can be written as $-mgz$, thus the action integral can be written $A = \int_0^2 \left(\frac{1}{2}m\dot{z}^2 + mgz\right) \mathrm{d}t$. Explicitly compute the action for the three proposed forms for $z(t)$ and show that the "real" answer gives the smallest value for this action integral.

5. Examine Figure 7.6 in your text; what is depicted is an "Atwood Machine" (you may remember these from PHYS 111). We will use the same setup described in the figure; two masses m_1 and m_2 are suspended by a massless inextensible string that passes over a pulley of radius R. We will use the variable x to describe the state of the system. Unlike the figure caption, however, we will *not* assume the pulley is massless. Though the pulley's axis is fixed (it isn't moving vertically), we will now state that the pulley has moment of inertia I. The string stays in contact with the pulley without slipping as it rotates.

   a) Write the new Lagrangian for this system (taking into account the kinetic energy associated with the pulley). As is always the case, we need you to write the entire Lagrangian in terms of x and known parameters (e.g. m_1, m_2, I) only.
   b) Use the Euler-Lagrange relations to find an expression for the acceleration $\ddot{x}$.
   c) Is this acceleration larger or smaller than the case solved in the text for a massless pulley?

6. A (stationary) triangular wedge with angle θ is placed on level ground on Earth. On top of this wedge, a disk with mass m, radius a, and moment of inertia I rolls down the wedge (without slipping).

   a) Find an expression for the acceleration of the center of mass of the disk $\ddot{x}$.
   b) Assume you roll (i) a sphere, (ii) a disk, and (iii) a ring down the same incline, all rolling without slipping. Place in order (from smallest to largest) $\ddot{x}_{\text{sphere}}$, $\ddot{x}_{\text{disk}}$ and $\ddot{x}_{\text{ring}}$ assuming all have the same mass and all have the same radius.

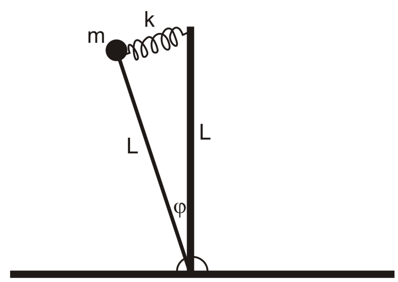


7. Examine the figure below. There is an unmovable vertical pole of length L connected via a massless spring with spring constant k to a second massless pole that has a point mass at its top of mass m. Assume the equilibrium length of the spring is essentially zero, so that the distance the string is stretched is equal to the distance between the top of the two poles. Use the angle φ as your generalized coordinate for this system.

   a) Write the Lagrangian for this system (in terms of m, L, k, g, and φ only).

   b) There is a stability point at $\varphi = 0$. Whether or not this is a stable equilibrium depends on the interplay between several of the material constants (e.g. m, k, etc.) associated with the problem. Write an inequality that must be satisfied if the equilibrium at $\varphi = 0$ is stable.

   c) Assuming the inequality in part (b) is satisfied, what is the angular frequency of small oscillations for this system?



8. Examine the figure below. There are two masses (m and M) that are connected to each other by a rigid rod of length d. Mass m and M are also connected to a pivot on the ceiling via rigid rods of length ℓ and L (respectively). This entire system is allowed to swing back and forth like a pendulum. Let the angle between ℓ and L be β (a constant) and the angle between ℓ and the ceiling be ϕ (your generalized coordinate for this problem).

   a) Write the Lagrangian for this system in terms of m, M, L, ℓ, g, β, ϕ, and $\dot{\phi}$ only.

   b) Use the Euler-Lagrange equations to write a differential equation for ϕ. This should be a second order differential equation, and you should write it in a the following form:

$$\ddot{\phi} + g(\phi) = 0$$

   (find $g(\phi)$).

   c) Find the equilibrium position for ϕ. [Hint. If $d = 0$, you should get $\phi = \pi/2$ for the equilibrium point]. Leave your answer in the form $\phi_\circ = \tan^{-1}(A + B)$ where A and B depend on m, ℓ, M, L, and β only.