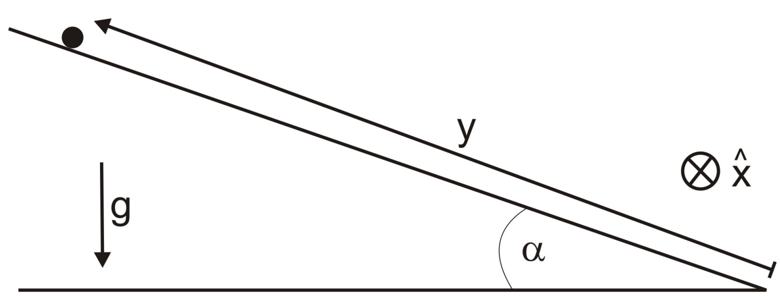
Assignments VIII and IX, PHYS 301 (Classical Mechanics)
Spring 2014
Due 3/21/14 at start of class

Homeworks VIII and IX both center on Lagrangian mechanics and involve many of the same skills. Therefore, I'm giving them both to you right now and having them due the same day (Friday, March 21st). These will be graded as two separate homework assignments (just like Homeworks VI and VII were); again, the horizontal line in the middle of the assignment indicates the break-point between the two assignments.

1. Let a point mass m move on a semi-infinite frictionless surface tilted at angle α with respect to the horizontal. Gravity points directly down. Let the variable y correspond to the distance from the vertex of the frictionless wedge as shown, and let $\hat{x}$ point into the page. (You may choose the origin of the x axis wherever you want). This is a system with two degrees of freedom; once you've chosen the origin of the x-axis, you can unique describe the position of the mass with two coordinates – x and y. (See picture below to try and clear up any ambiguity).

   a) Write the Lagrangian of this system in terms of variables x and y.
   b) Find the differential equations governing $x(t)$ and $y(t)$.
   c) Solve the differential equation for $x(t)$ (you may have a couple of undetermined constants).
   d) Solve the differential equation for $y(t)$ (again, you may have a couple of undetermined constants).
   e) If $x(t=0) = x_\circ$, $\frac{\mathrm{d}x}{\mathrm{d}t}|_{t=0} = v_{x0}$, $y(t=0) = y_\circ$, and $\frac{\mathrm{d}y}{\mathrm{d}t}|_{t=0} = v_{y0}$, write expressions (without arbitrary constants) for $x(t)$ and $y(t)$.



2. We did problem 4.36 earlier this semester in class. Go back to that problem and the associated figure (figure 4.27). We mentioned that this system only has 1 degree of freedom. Find the Lagrangian for this system, using θ as the one coordinate. (Trigonometry will be of some help here). Do *not* apply the Euler-Lagrange equations and try to find $\theta(t)$. (It is all kinds of horrible.) Just find the Lagrangian.

3. Figure 7.17 in your text shows a wire bent into the shape of a parabola that rotates about the z axis with constant angular velocity ω. On this wire, a mass m slides (without friction). This system presumably exists on Earth so that there is gravity pointing in the $-\hat{z}$ direction. We're going to have you examine a system *similar*, but not identical to this. Instead of a bead moving along a parabola, it moves along a wire that can be described by the curve $z = ks^4$ where s is the distance from the z axis. As in figure 7.17, the system rotates with constant angular velocity ω about the z axis.

   a) Write down the Lagrangian for this system in terms of s as the generalized coordinate. (Of course, you can also use parameters of the problem like m, k, ω. What you may *not* do is include z in your Lagrangian)

   b) Find any points of equilibrium (if they exist).

4. Examine Figure 7.6 in your text; what is depicted is an "Atwood Machine" (you may remember these from PHYS 111). We will use the same setup described in the figure; two masses m_1 and m_2 are suspended by a massless inextensible string that passes over a pulley of radius R. We will use the variable x to describe the state of the system. Unlike the figure caption, however, we will *not* assume the pulley is massless. Though the pulley's axis is fixed (it isn't moving vertically), we will now state that the pulley has moment of inertia I. The string stays in contact with the pulley without slipping as it rotates.

   a) Write the new Lagrangian for this system (taking into account the kinetic energy associated with the pulley). As is always the case, we need you to write the entire Lagrangian in terms of x and known parameters (e.g. m_1, m_2, I) only.

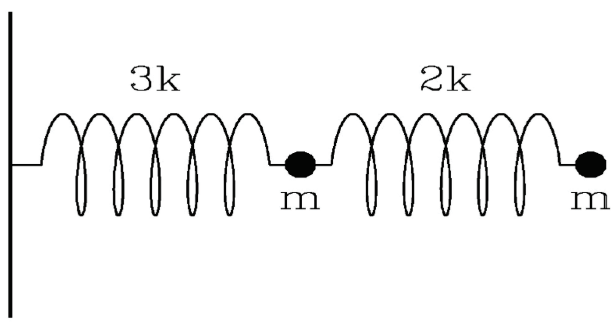
   b) Use the Euler-Lagrange relations to find an expression for the acceleration $\ddot{x}$.

   c) Is this acceleration larger or smaller than the case solved in the text for a massless pulley?

5. The two massless springs shown below are constrained to move along a single horizontal direction. Choose as generalized coordinates the displacements of the springs from their equilibrium lengths. (Let the left-hand spring have equilibrium length ℓ_1 and the right hand spring have equilibrium length ℓ_2. Therefore, the left-hand mass is at position $\ell_1 + x_1(t)$ where x_1 is the first of two generalized coordinates you need. (I'll let you figure out how you need to define $x_2(t)$ so that it describes the displacement of the second spring from its equilibrium length – make sure you think about this carefully). Since no vertical motion occurs, you may neglect gravity for this problem.

   a) Write the Lagrangian for the system.

   b) Use the Euler-Lagrange equations to find two differential equations that describe the motion. (They will be coupled second order differential equations. No need to solve them at this point, but if you want to show off, go ahead and do it. We'll talk about systems like this in a couple more chapters).



6. Figure 7.14 in your text (pg. 286) shows a simple pendulum of mass m and length ℓ whose point of support is attached to the edge of a wheel of radius R rotating at a fixed angular velocity ω. At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and show the equation of motion takes the form $\ell\ddot{\phi} = -g\sin\phi + \omega^2 R\cos(\phi - \omega t)$. Confirm this makes sense for $\omega = 0$.

7. Do problem 7.44 in your text. (The previous problem is essentially problem 7.29).

8. Similar to the above, replot $\phi(t)$ with the new conditions with $g = \ell = 1$ but $\omega = 3$ and $R = 2$. Let initial conditions be $\phi = 2$ and $\dot{\phi} = 0$. Plot your solution for $0 < t < 10$. Comment on your plot.

---

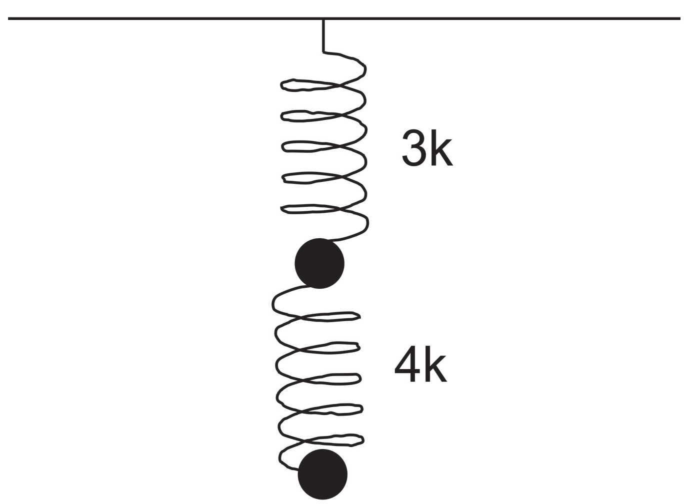
9. Below is a picture of two massless springs connected to the ceiling. The equilibrium (un-stretched) length of the top spring is ℓ_1 and the equilibrium (un-stretched) length of the bottom spring is ℓ_2. The top spring has spring constant $3k$ and the bottom spring has spring constant $4k$. The mass between the two springs when aligned as shown has mass m_1 and the mass connected to the bottom of the bottom spring has mass m_2. This system is near the surface of the earth (where the gravitational acceleration is g downward) and the ceiling is level. The masses are constrained to move in the vertical direction *only*.

   a) Let x_1 be the distance from the ceiling to the top mass when aligned as shown in the figure. (This will be longer than ℓ_1 at equilibrium due to the fact that the two masses below it help extend it beyond its un-stretched length). Let x_2 be the distance from the ceiling to the bottom mass when aligned as shown in the figure. Find the equilibrium (stable) values for x_1 and x_2. (You may use any method you wish).

   b) Let $s_1 = x_1 - \ell_1$ and $s_2 = x_2 - x_1 - \ell_2$ be your generalized coordinates. Write the Lagrangian for this system in terms of s_1 and s_2 (You don't have to complete part (a) successfully to do this). (Be careful).

   c) Use the Euler-Lagrange Equations to find coupled differential equations for $\ddot{s}_1$ and $\ddot{s}_2$.

   d) Let $m_2 = 0$. This effectively uncouples things, and makes it rather easy to solve one of the differential equations. (i) Find $s_2(t)$ and briefly explain why this makes sense. (ii) Use your answer from part (a) to find an equilibrium length for s_1 when $m_2 = 0$. Show that this constant value of $s_1(t) = x_{1\text{equil}} - \ell_1$ satisfies the remaining differential equation for $\ddot{s}_1$ found in part (c) once the simplification $m_2 = 0$ has been made.

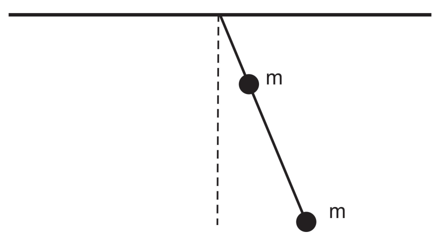


10. Look back at the inclined plane shown in the figure to problem 1. This problem uses a similar (though not identical) geometry. In this system, we will be changing the inclined plane over time, so that $\alpha = \omega t$ for $t \geq 0$ with ω some positive constant.

    a) Neglect the x direction entirely and treat this as a one-dimensional system with y as your generalized coordinate. Show that $\ddot{y} - \omega^2 y = -g \sin \omega t$.
    b) Solve the above differential equation assuming that $y(t=0) = y_\circ$ and the initial velocity is 0.

11. A rigid pendulum of length L has a mass m on its end and another mass m somewhere on the support between the pivot and the end at a distance ℓ from the pivot. Use ϕ (the angle from the vertical) as your generalized coordinate. Note that ℓ is *not* a generalized coordinate; it is a constant (and can be assumed to be known; ℓ may appear in your final answers!).

    a) Write the Lagrangian for this system.
    b) Use the Euler-Lagrange Equations to find a homogeneous differential equation for $\ddot{\phi}$.
    c) For small angles, find the angular frequency of small oscillations.
    d) Find the value of ℓ between 0 and L that maximizes or minimizes this frequency of small oscillations. (Which is it; a max or a min?)



12. Consider a normal pendulum consisting of a mass m attached to a massless string of length ℓ. After the pendulum is set into motion, the length of the string is shortened at a constant rate $\frac{d\ell}{dt} = -\alpha$ where α is a positive constant. The point of suspension remains fixed.

    a) Find the Lagrangian for this system in terms of the single generalized coordinate θ and constants.
    b) If you use the Euler-Lagrange equation to obtain a differential equation for θ, you should be able to make this system look remarkably like a damped harmonic oscillator (for small angles so that $\sin \theta \approx \theta$). To remind you, the equation for a damped harmonic oscillator can be written $\ddot{x} + 2\beta\dot{x} + \omega_\circ^2 x = 0$. The differential equation you get here looks exactly like this with $x \leftrightarrow \theta$. What would β and $\omega_\circ$ be in terms of the variables in this problem to make the analogy complete?

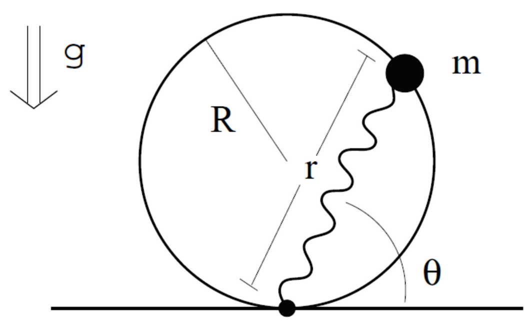
MORE ON BACK!

13. A bead of mass m slides without friction on a fixed vertical hoop of radius R. The bead moves under the combined action of gravity and a spring attached to the bottom of the hoop. *For simplicity, we assume the equilibrium length of the spring is zero*, so the force due to the spring is $-kr$ where r is the instantaneous length of the spring. Assume that the bead can pass unhindered through the point at the bottom of the loop.

   a) Write down the Lagrangian for the problem, using θ as drawn as the generalized coordinate. (Note; that means you'll have to be clever and find r as a function of θ; since R is a constant in the problem it is allowed to show up in your final expression. r may not.)

   b) Use the Euler-Lagrange equation to come up with the general equation of motion for the problem (in terms of θ and its time derivatives). Simplify so that the $\ddot{\theta}$ term has a coefficient of 1.

   c) Show that the angular frequency of small oscillations about the bottom of the loop is $\left(\frac{g}{R} + \frac{k}{m}\right)^{1/2}$.



14. A body is released from a height of 20 meters (in a vacuum) and approximately 2 seconds later it strikes the ground. The equation for the distance of fall z during time t could conceivably have any of the following forms (where g has different units in the three expressions):

$$z = gt \qquad z = \frac{1}{2}gt^2 \qquad z = \frac{1}{4}gt^3$$

all of these yield $z = 20$ meters for $t = 2$ seconds. Show (via direct computation for the three cases) that the correct form gives the smallest value for the integral of the Lagrangian.