Assignment VIII, PHYS 272 (MAP)
Fall 2014
Due 10/31/14 at start of class

1. The differential equation for a particle falling vertically under the presence of air-resistance proportional to velocity βv can be written as:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - \beta v$$

   where we have chosen v to be positive in the downward direction and β is a constant associated with the physical properties of the system.

   a) What are the units of β?
   b) What is the steady-state value (e.g. free-fall speed) of the system? (Hint...what does steady-state mean in terms of v?)
   c) If a particle of mass m is dropped (from rest), find $v(t)$. (In other words, solve this differential equation under the initial condition $v_{t=0} = 0$.) A change of variable might be helpful, but I want your answer in terms of variables given in the problem.

2. The above problem was a simplified version of the truth. In reality, a freely falling body of mass m near Earth's surface encounters an air resistance equal to $\beta v + \alpha v^2$ where v is the speed.

   a) Set up a new, modified differential equation governing $v(t)$.
   b) Without solving the differential equation, find the limiting velocity (e.g. $v(t \to \infty)$).
   c) Show that this expression reduces to the expected value obtained in problem 1b when $\alpha \to 0$.

3. A particle of mass m moves horizontally along the x axis subject to a resisting force of the form ηv^3. Find the velocity in terms of the time t and the initial velocity $v_\circ$.

4. The rocket equation can be written as follows:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = 0$$

   Here, m is a function of t (it is the changing mass of the rocket), v is the velocity of the rocket (also changing with respect to time), and u is the speed of the rocket exhaust with respect to the rocket (which is a *constant*). Assume this rocket starts from rest with some mass $m_\circ$.

   a) Find $v(t)$ in terms of $m(t)$, $m_\circ$, and u.
   b) You will find that your answer to part (a) above is larger when the ratio $m_\circ/m$ is maximized. Let's say you have the choice of adding some small mass δm to $m_\circ$ or subtracting the same small mass δm from $m(t)$. Which one would have resulted in a larger $v(t)$? Explain your answer/reasoning.

5. The momentum p of an electron at speed v near the speed of light c increases via the formula $p = \gamma mv = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$ where m is a constant (the mass of the electron). If an electron is subject to a constant force F, Newton's second law describing its motion is:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right) = F$$

   a) If you solve this differential equation, you should find that the answer for $v(t)$ takes the form:

$$v(t) = \frac{c(Ft+A)^2}{g(A,t,F,m,c)}$$

   Where $g(A,t,F,m,c)$ is some function of variables t, F, m, c and the same integration constant that appears in the numerator. Find $g(A,t,F,m,c)$. Hint: what is $\int \mathrm{d}(\text{mess})$?

   b) Show that $v(t) \to c$ as $t \to \infty$.

   c) Find the distance traveled by the electron in time t if it starts from rest. *Do not use Mathematica or other computer software/assistance!* The following (indefinite) integral may come in handy: $\int \frac{t\mathrm{d}t}{(a^2+t^2)^{1/2}} = (a^2+t^2)^{1/2}$. ($a$ is any constant). Choose your integration constant so that $x(t=0) = 0$.

   d) Show that your expression for c reduces to the expected/classical value $\Delta x = \frac{1}{2}\left(\frac{F}{m}\right)t^2$ for small t. (Hint: Binomial expansion!)

6. A body at a temperature of T_i is placed in a room of unknown temperature. (The room temperature is taken to be constant for all time). If you didn't know this, the time rate of change of the temperature of a cooling body is proportional to the temperature difference between the body and its surroundings).

   a) Develop a differential equation governing $T(t)$ (the temperature as a function of time) for the body if the proportionality constant is taken to be k and the temperature of the room is $T_\circ$. Make sure your differential equation makes sense so that when k is positive, an object heats up if its surroundings are at a higher temperature and cools down of its surroundings are at a colder temperature.

   b) Solve the differential equation you developed in part (a).

   c) Verify that as $t \to \infty$, $T(t) \to T_\circ$ and as $t \to 0$, $T(t) \to T_i$.

   d) Plug your solution from part (b) back into the differential equation developed in part (a) to verify it works.

7. One of the most persistent partial differential equations you will see in undergraduate Physics is the Laplace Equation, which takes the form:

$$\nabla^2 V = 0$$

Where V is a function of the three spatial coordinates. This equation is solvable in cartesian, cylindrical, and spherical coordinates by a method known as "Separation of Variables" which, if we don't talk about in our class, you will see in E&M, Quantum Mechanics, and probably a couple other places as well. The basic idea is to assume V is a function of the spatial variables in a very particular way; in particular, we assume:

$$\begin{aligned} V(x,y,z) &= X(x)Y(y)Z(z) \\ V(s,\phi,z) &= S(s)\Phi(\phi)Z(z) \\ V(r,\theta,\phi) &= R(r)\Theta(\theta)\Phi(\phi) \end{aligned}$$

where capital letters indicate unspecified functions of the associated variable. After some manipulation (that you'll see either in our class, or in your upper level classes) you will find that you can take the PDE $\nabla^2 V = 0$ and obtain 3 different 2nd order ordinary differential equations (not partial differential equations) that are a bit friendlier to our usual methods. Below, I give the differential equations for two of the coordinates you get for two of the different coordinate systems (some of the other ones involve some "special functions" you may not have seen yet). I also supply the proposed solution to each of the differential equations (don't worry at this point how we attained each of these). Your task in this problem is to explicitly verify that the proposed solution to the supplied differential equation works. (E.g. "plug and chug" and demonstrate that the solution does, in fact, satisfy the specified differential equation).

a) Cartesian Coordinates:

   i) Differential Equation:

   $$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -k^2 X$$

   Proposed Solution:

   $$X = A\sin(kx) + B\cos(kx)$$

   with A and B arbitrary constants.

   ii) Differential Equation:

   $$\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = k^2 Y$$

   Proposed Solution:

   $$Y = C\sinh(ky) + D\cosh(ky)$$

   with C and D arbitrary constants.

b) Spherical Coordinates

i) Differential Equation:

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \ell(\ell+1)$$

Proposed Solution:

$$R = Er^{\ell} + \frac{F}{r^{\ell+1}}$$

with E and F arbitrary constants.

ii) Differential Equation:

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -m^2$$

Proposed Solution:

$$\Phi = G\mathrm{e}^{im\phi} + H\mathrm{e}^{-im\phi}$$

with G and H arbitrary constants.