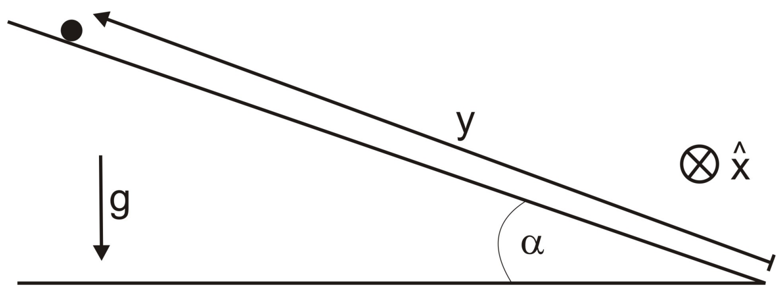
Assignment VIII, PHYS 301 (Classical Mechanics)
Spring 2015
Due 3/20/15 at start of class

1. Let a point mass m move on a semi-infinite frictionless surface tilted at angle α with respect to the horizontal. Gravity points directly down. Let the variable y correspond to the distance from the vertex of the frictionless wedge as shown, and let $\hat{x}$ point into the page. (You may choose the origin of the x axis wherever you want). This is a system with two degrees of freedom; once you've chosen the origin of the x-axis, you can uniquely describe the position of the mass with two coordinates – x and y. (See picture below to try and clear up any ambiguity).

   a) Write the Lagrangian of this system in terms of variables x and y.
   b) Find the differential equations governing $x(t)$ and $y(t)$.
   c) Solve the differential equation for $x(t)$ (you may have a couple of undetermined constants).
   d) Solve the differential equation for $y(t)$ (again, you may have a couple of undetermined constants).
   e) If $x(t=0) = x_\circ$, $\frac{\mathrm{d}x}{\mathrm{d}t}\big|_{t=0} = v_{x\circ}$, $y(t=0) = y_\circ$, and $\frac{\mathrm{d}y}{\mathrm{d}t}\big|_{t=0} = v_{y\circ}$, write expressions (without arbitrary constants) for $x(t)$ and $y(t)$.



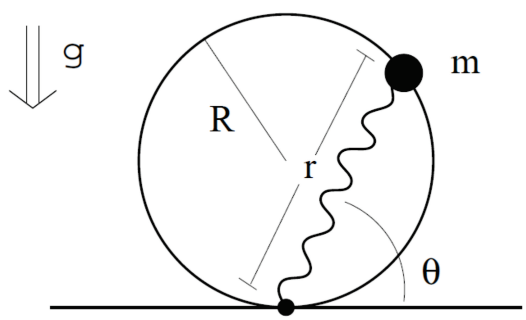
2. Look back at the inclined plane shown in the figure to problem 1. This problem uses a similar (though not identical) geometry. In this system, we will be changing the inclined plane over time, so that $\alpha = \omega t$ for $t \geq 0$ with ω some positive constant. (The incline gets steeper over time).

   a) Neglect the x direction entirely and treat this as a one-dimensional system with y as your generalized coordinate. Show that $\ddot{y} - \omega^2 y = -g \sin \omega t$.
   b) Solve the above differential equation assuming that $y(t=0) = y_\circ$ and the initial velocity is 0.

3. (The following is problem 7.18 from your text). A mass m is suspended from a massless string, the other end of which is wrapped several times around a horizontal cylinder of radius R and moment of inertia I, which is free to rotate about a fixed horizontal axle. Using a suitable coordinate, set up the Lagrangian and the Lagrange equation of motion, and find the acceleration of the mass m. [The kinetic energy of the rotating cylinder is $\frac{1}{2}I\omega^2$].

4. A bead of mass m slides without friction on a fixed vertical hoop of radius R. The bead moves under the combined action of gravity and a spring attached to the bottom of the hoop. *For simplicity, we assume the equilibrium length of the spring is zero*, so the force due to the spring is $-kr$ where r is the instantaneous length of the spring. Assume that the bead can pass unhindered through the point at the bottom of the loop.

   a) Write down the Lagrangian for the problem, using θ as drawn as the generalized coordinate. (Note; that means you'll have to be clever and find r as a function of θ; since R is a constant in the problem it is allowed to show up in your final expression. r may not.)
   b) Use the Euler-Lagrange equation to come up with the general differential equation of motion for the problem (in terms of θ and its time derivatives). Simplify so that the $\ddot{\theta}$ term has a coefficient of 1.
   c) Show that the angular frequency of small oscillations about the bottom of the loop is $\left(\frac{g}{R} + \frac{k}{m}\right)^{1/2}$.



5. The hot-air balloon pendulum! A hot air balloon accelerates vertically with constant acceleration a near the surface of the earth. On the bottom of the balloon's passenger basket, someone cleverly mounted a simple pendulum with arm-length ℓ and bob-mass m. You may assume that the arm is massless.

   a) Form the Lagrangian for this system (assuming you stay close to the surface of the Earth). (The Lagrangian will depend on t and the angle ϕ with respect to the vertical, as well as constants).
   b) Find $\phi(t)$ for small oscillations.
   c) Rewrite the Lagrangian for this system without the assumption that you stay close to the Earth. (i.e. g is no longer a constant!) (Don't worry; I won't make you solve it).

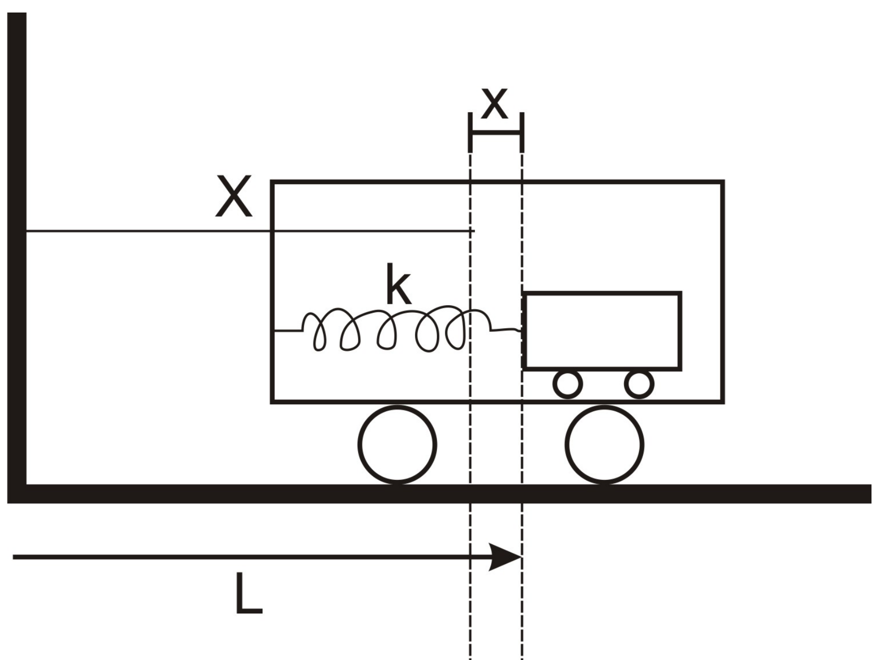
6. Examine the figure below. A cart of mass m is mounted inside a larger cart of irrelevant mass. The two carts are attached via a spring (with spring constant k) as shown. Let the distance the small cart is displaced *with respect to the large cart* be x. (In other words, when $x = 0$, the spring is neither compressed nor extended). Let X be the distance from a fixed point (say a wall) to the point where $x = 0$. Thus, the position of the small cart with respect to the same wall would be $L = X + x$.

   The large cart is exposed to a forced oscillation in such a way that $X = A\cos(\omega t)$ with ω and A constant. (Don't worry about the mechanism forcing this oscillation – if you want, just envision someone applying a sinusoidal motion to the big cart by hand).

   a) Set up the Lagrangian and use it to find a differential equation for x. Your answer should only have A, ω, k, m, t, and (of course) x and its derivatives involved.

   b) The differential equation you found in part (a) should look really familiar. Use the particular solution of that differential equation (found in HW 6, though you could find it again here if necessary) to describe (in words) the behavior of $x(t)$ if $\omega \ll \sqrt{\frac{k}{m}}$. (Comment on the amplitude and phase of $x(t)$ compared to $X(t)$).

   c) Do the same as part (b) except describe the situation if $\omega \gg \sqrt{\frac{k}{m}}$

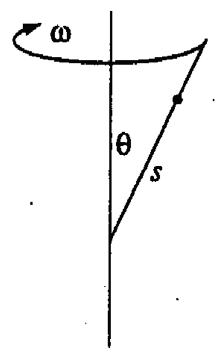


(More on back!)

7. A bead is constrained to slide on a frictionless rod that is fixed at an angle θ with a vertical axis and is rotating with angular frequency ω about the axis, as shown below.

   a) Take the distance s along the rod as the variable. What is the Lagrangian for the bead? (Assume you are on Earth with gravity pointing down).

   b) Use your Lagrangian found in part (a) to write a differential equation for s. (As a check on your answer, in class we did the case where $\theta = \pi/2$). Note: You do not need to solve this differential equation, just find it!



8. A uniform sphere of radius a and mass M is constrained to roll without slipping (don't forget the rotational energy!) on the lower half of the inner surface of a hollow cylinder of inside radius R. You may assume the cylinder remains stationary with respect to Earth. (In particular, the cylinder is nailed to the ground and does not roll).

   a) Determine the Lagrangian function.

   b) Write down the equation of motion. (You do not need to solve the differential equation; just get the differential equation).

   c) What is the frequency (NOT angular frequency) of small oscillations?