Assignment IX, PHYS 301 (Classical Mechanics)
Spring 2015
Due 4/3/15 at start of class

1. Show that if a star of mass M were surrounded by a spherical dust cloud of uniform mass density $\rho_\circ$, the gravitational field within the dust cloud would be:

$$\vec{g} = -\left(\frac{MG}{r^2} + \frac{4\pi\rho_\circ G r}{3}\right)\hat{r}$$

where $\vec{r}$ is a vector from the center of the star to any point in the dust cloud.

2. Assume the Earth is a uniform sphere with density ρ.

   a) Show that a particle dropped into a straight hole (of negligible width) passing through Earth's center would execute simple harmonic motion.

   b) Find the period of the oscillation of the particle described in part (a). [Your answer should only depend on Earth's density, ρ, not its size!]

   c) Recall (or look up, if you have to) the Earth's radius and mass. Use this information to numerically compute what the frequency of oscillation of the particle in parts (a) and (b) would have to be. (You may use a calculator).

3. A small hole of negligible diameter is bored along a radius into a sphere whose density ρ at a given point is a function of r, the distance of the point from the center of the sphere. It is found that the gravitational force exerted on the particle in the hole is independent of the distance of the particle from the center of the hole. If the total mass of the sphere is M and its total radius is R, what is $\rho(r)$?

4. Assume that the density of a star is a function of the distance r measured from the center of the star and is given by:

$$\rho(r) = \frac{Ma^2}{2\pi r\left(r^2+a^2\right)^2}, \qquad 0 \le r < \infty$$

where M is the mass of the star and a is a constant that helps to determine the size of the star.

   a) Find the gravitational field as a function of r. (No calculators or Mathematica! You can do this associated integral without technology!)

   b) Find the gravitational potential as a function of r. (Hint: $\Phi(\vec{r}) = -\int_\infty^r \vec{g}\cdot \mathrm{d}\vec{r}$). (Another hint: $\int \frac{\mathrm{d}x}{(1+x^2)} = \tan^{-1}(x) + C$).

5. The orbit of a particle moving in a central field is a circle passing through the origin and obeying the equation $r = r_\circ \cos\theta$. The underlying force law is of the form $F(r) \propto r^n$. Find n. (Hint: It is an integer. Second hint: it is not -3).

6. A particle moves in a spiral orbit given by $r = a\theta$ (with a a constant). If $\theta = kt^1$ (with k constant), is the force a central force? If so – how do you know? If not, then $\theta = kt^\xi$ (with k constant and $\xi \neq 1$) DOES correspond to a central force for some value of ξ (still with the spiral orbit described by $r = a\theta$). Find the value of ξ.

7. A point mass moves in a potential of the form $U(r) = \frac{-C}{3r^3}$ with C some constant.

   a) Given ℓ, find the maximum value of $U_{\text{eff}}(r)$. Your answer should be in terms of ℓ, C, and m only.

   b) Let a particle come in from infinity with initial speed $v_\circ$ and impact parameter b. In terms of C, m, and $v_\circ$, there exists some maximal value of b (call it b_{max}) for which the particle is just barely captured by the potential. Your goal is to find the "cross-section" σ for capture in this potential. $\sigma = \pi b_{\text{max}}^2$, so you will need to first find b_{max}. [Recall, the "impact parameter" b of a trajectory is defined to be the closest distance to the origin the particle would achieve if it moved in the straight line determined by its initial velocity far from the origin.]

8. A comet moves in a parabolic orbit lying in the plane of the Earth's orbit. Regarding the Earth's orbit as circular and of radius a, show that the points where the comet intersects the Earth's orbit are given by:

$$\theta = \pm \cos^{-1}\left(\frac{2p}{a} - 1\right)$$

   where p is the perihelion distance of the comet defined at $\theta = 0$.