Assignment IX, PHYS 301 (Classical Mechanics)
Spring 2017
Due 4/14/17 at start of class

1. An incompressible but moldable blob of matter of total mass M is to be situated between two flat horizontal planes at $z = 0$ and $z = 10$ so that the moment of inertia around the z axis is as small as possible. What shape should the blob take?

2. A thin (pseudo-1-dimensional) rod of length L lies between the origin and $L\hat{x}$. The linear mass density of this rod can be written as:

$$\lambda(x) = \lambda_\circ \left(1 + 2\frac{x^3}{L^3}\right)$$

   a) What is the total mass of the rod (in terms of $\lambda_\circ$ and L)?

   b) What is the moment of inertia of the rod with respect to the $\hat{y}$ axis going through the origin? Leave your answer in terms of the Mass of the rod M and the length of the rod L ONLY.

   c) What is the moment of inertia of the rod with respect to an axis parallel to the $\hat{y}$ axis that goes through the point $L\hat{x}$? (Again leave your answer in terms of the Mass of the rod M and the length of the rod L only).

   d) Where is the center of mass of the rod?

   e) The rod is balanced on its lighter end on a rough floor and then tips over; the point in contact with the Earth doesn't move (it pivots about its end). What is the rod's angular velocity when it hits the floor? (You may neglect the rotation of the Earth). (Leave your answer in terms of g and L only.)

   f) If the rod was balanced on its heavy end and the same thing happened as in part (e) above, would the angular velocity be larger or smaller when the rod hits the floor? Use a calculation to justify your answer.

3. A hemisphere of radius a has density $\rho = \alpha r^2$ for $0 \leq r \leq a$ and $0 \leq \theta \leq \pi/2$ (All 2π values of ϕ are permitted).

   a) Find the moment of inertia with respect to the $\hat{z}$ axis (going through the origin). (Take care – you may want to consider cylindrical coordinates. It can be done in spherical, but you have to be careful).

   b) The "radius of gyration" is a distance. The radius of gyration for an object of mass M is defined as "the distance from the axis of rotation that a *point mass* would have to be so that it would have the same moment of inertia as the object's moment of inertia with respect to the same axis". In equation form, the radius of gyration (κ) is often written as $\kappa = \left(\frac{I}{M}\right)^{1/2}$. Find the radius of gyration for this hemisphere with respect to an axis parallel to the $\hat{z}$ axis going through the cartesian point $a\hat{x}$.

4. Let a rectangular plate centered at the origin in the $x-y$ plane have dimensions $2a$ x $2b$. ($-a < x < a$ and $-b < y < b$). Let the mass density per unit area of the plate be described by:

$$\sigma(x,y) = \begin{cases} \alpha x^2 + c & -a < x < a \qquad \text{and} \qquad -b < y < b \\ 0 & \text{otherwise} \end{cases}$$

   a) Find the units for α and c.
   b) Find the total mass of the plate (in terms of α, a, b, and c).
   c) Find the moment of inertia with respect to an axis in the $\hat{z}$ direction through the origin.
   d) Let $\alpha = 0$ and show that your answer to part (c) reduces to that expected for a uniform rectangular plate with mass per unit area c.
   e) Show that the moment of inertia with respect to an axis in the $\hat{z}$ direction that goes through the point $a\hat{x} + b\hat{y} + 0\hat{z}$ is equal to:

$$I = \frac{16}{3}ab\left[\alpha a^2\left(\frac{2}{5}a^2 + \frac{1}{3}b^2\right) + c(a^2+b^2)\right]$$

5. A rod of length $2b$ runs along the x-axis from the point $-b\hat{x}$ to the point $b\hat{x}$. The ends of this rod mark the centers of two spheres, each of radius $a < b$. (So there's a sphere of radius a centered at each of the two points $\pm b\hat{x}$). (This system looks like an old-school "barbell"). Let the rod have linear mass density λ and each sphere has volumetric mass density ρ. By construction, the center of mass is at the origin no matter what ρ and λ are. For this system:

   a) Find the principal moments of inertia about the center of mass. (You may look up the moments of inertia of each component to save you some work if you'd like).
   b) Find the ratio of I_z/I_x for $\lambda \to 0$.
   c) Introductory Physics texts often give the following simple approximation for the nuclear radius: $r \approx (1.2 \times 10^{-15}\text{ m})A^{1/3}$ where A is the number of nucleons in the atom. The bond-length of an N_2 molecule is approximately 145×10^{-12} m. Assuming the bond has no mass, numerically calculate I_z/I_x for an N_2 molecule using the basic model proposed above. Interpret/comment on your result. (You may use a calculator or software to aid you).

6. Three point masses represent a rigid body; the position/magnitude of the masses are as follows:

$$\begin{cases} m_1 = m & 2a\hat{y} + a\hat{z} \\ m_2 = m & a\hat{y} + 2a\hat{z} \\ m_3 = 2m & a\hat{x} \end{cases}$$

   a) Write down the full Inertia tensor I for this system. (Actually compute each entry in terms of m and a).
   b) Find the principle moments of inertia for this system.
   c) Find the orthogonal principle axes for this rotation and explicitly verify they are orthogonal. [How do you show that three axes are mutually orthogonal? Either: (i) show $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$ or (ii) show $\vec{v}_1 \times \vec{v}_2 = \lambda \vec{v}_3$ with λ some real constant.]