Assignment X, PHYS 301 (Classical Mechanics)
Spring 2015
Due 4/10/15 at start of class

1. Examine figure 7.9 on page 261 of your text. A bead of mass m is threaded on a frictionless wire hoop of radius R and aligned as shown, rotated with constant angular velocity ω. We are going to modify this system slightly. In addition to the drawn figure, imagine that there is a massless spring (with spring constant k and equilibrium length zero) connected between the bead and the top of the hoop. Find the equilibrium angle(s) for θ in this modified system.

2. Again, starting with the figure on page 261 of your text, we will now add two springs to the system; first, a massless spring with spring constant k and of equilibrium length zero is connected between the bead and the top of the hoop. The second string also is massless, also has spring constant k, and also has equilibrium length zero and is connected between the bead and the bottom of the hoop. Find the equilibrium angle(s) for θ.

3. Let a rectangular plate centered at the origin in the $x-y$ plane have dimensions $2a$ x $2b$. ($-a < x < a$ and $-b < y < b$). Let the mass density per unit area of the plate be described by:

$$\sigma(x,y) = \begin{cases} \alpha x^2 + c & -a < x < a \qquad \text{and} \qquad -b < y < b \\ 0 & \text{otherwise} \end{cases}$$

   a) Find the units for α and c.
   b) Find the total mass of the plate.
   c) Find the moment of inertia with respect to an axis in the $\hat{z}$ direction through the origin.
   d) Let $\alpha = 0$ and show that your answer to part (c) reduces to that expected for a uniform rectangular plate with areal mass density c.
   e) Show that the moment of inertia with respect to an axis in the $\hat{z}$ direction that goes through the point $a\hat{x} + b\hat{y} + 0\hat{z}$ is equal to:

$$I = \frac{16}{3}ab\left[\alpha a^2\left(\frac{2}{5}a^2 + \frac{1}{3}b^2\right) + c(a^2 + b^2)\right]$$

4. A hemisphere of radius a has density $\rho = \alpha r^2$ for $0 \le r \le a$ and $0 \le \theta \le \pi/2$ (All 2π values of ϕ are permitted).

   a) Find the moment of inertia with respect to the $\hat{z}$ axis (going through the origin). (Take care – you may want to consider cylindrical coordinates. It can be done in spherical, but you have to be careful).
   b) The "radius of gyration" is a distance. The radius of gyration for an object of mass M is defined as "the distance from the axis of rotation that a *point mass* would have to be so that it would have the same moment of inertia as the object's moment of inertia with respect to the same axis". In equation form, the radius of gyration (κ) is often written as $\kappa = \left(\frac{I}{M}\right)^{1/2}$. Find the radius of gyration for this hemisphere with respect to an axis parallel to the $\hat{z}$ axis going through the cartesian point $a\hat{x}$.

5. A rod of length $2b$ runs along the x-axis from the point $\langle -b, 0, 0\rangle$ to the point $\langle b, 0, 0\rangle$. The ends of this rod mark the midpoints of two spheres, each of radius $a < b$. (So there's a sphere of radius a centered at each of the two points $\langle \pm b, 0, 0\rangle$). (This system looks like an old-school "barbell"). Let the rod have linear mass density λ and each sphere has volumetric mass density ρ. By construction, the center of mass is at the origin no matter what ρ and λ are. For this system:

   a) find the principal moments of inertia about the center of mass. (You may look up the moments of inertia of each component to save you some work if you'd like).

   b) Find the ratio of I_z/I_x for $\lambda \to 0$.

   c) Introductory Physics texts often give the following simple approximation for the nuclear radius: $r \approx (1.2 \times 10^{-15}\text{ m})A^{1/3}$ where A is the number of nucleons in the atom. The bond-length of an N_2 molecule is approximately 145×10^{-12} m. Assuming the bond has no mass, numerically calculate I_z/I_x for an N_2 molecule using the basic model proposed above. Interpret/comment on your result. (You may use a calculator or software to aid you).

6. Three point masses represent a rigid body; the position/magnitude of the masses are as follows:

$$\begin{aligned} m_1 &= m & \langle 0, 2a, a\rangle \\ m_2 &= m & \langle 0, a, 2a\rangle \\ m_3 &= 2m & \langle a, 0, 0\rangle \end{aligned}$$

   a) Write down the full Inertia tensor I for this system.

   b) Find the principle moments of inertia for this system.

   c) Find the orthogonal principle axes for this rotation and explicitly verify they are orthogonal. [How do you show that three axes are mutually orthogonal? Either: (i) show $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$ or (ii) show $\vec{v}_1 \times \vec{v}_2 = \lambda \vec{v}_3$ with λ some real constant.]