Homework 10, PHYS 415 (Fluid Mechanics)
Spring 2019
Due Thursday 4 April 2019 at Beginning of Class

As always, turn in your legible and annotated work on separate paper. Note the due date!

1. A velocity field in cylindrical coordinates is written $v_s = 0$, $v_\varphi = asz$, $v_z = 0$ where a is a constant.

   a) Find the components of the vorticity ω_s, ω_φ, and ω_z.

   b) Verify that $\vec{\nabla} \cdot \vec{\omega} = 0$.

2. In class, we found the velocity field associated with a Rankine vortex. Here, we will have you find the velocity field for something similar to two co-axial Rankine vortices. Consider the following vortex field:

$$\vec{\omega} = \begin{cases} \omega_\circ \hat{z} & s \leq R \\ -3\omega_\circ \hat{z} & R < s \leq 4R \\ 0 & s > 4R \end{cases}$$

   a) Find $v_\varphi(s)$ for all s.

   b) Use your answer to part (a) to find all radial distances in the flow where the fluid is stationary.

   c) Calculate $\Gamma(s)$.

   d) Use a computer program to make a plot of v_φ as a function of s. Let $R = 1.5$ m and $\omega_\circ = 3.5\ \mathrm{s}^{-1}$. Show the domain $0 \leq s \leq 30$ m.

3. I haven't done as good of a job as I had hoped in keeping the homework astrophysically relevant for those of you who are Astrophysics majors. Although I haven't asked any astronomers for sure to determine if this problem is relevant to your work, I have a hunch it might be. Don't let the massive amount of text fool you – I don't think the solution to this one is particularly gnarly.

   In the vorticity video shown in class before break, we saw the creation of two different vortices being shed off the trailing edge of a wing-like geometry; those vortices were oriented with their rotation opposite to each other and had equal magnitudes and we saw that they moved together down the screen. This problem deals with two different vortices aligned with the same sense of rotation (both rotating, say, counterclockwise) with their centers separated by some distance h. Let us model each of these vortices as a Rankine vortex with very small radius $R \ll h$.

   We will assume rotation outside this small radius R is irrotational (say because the underlying fluid is inviscid); in this setup, the flow at a point can be obtained by superposing the velocity field induced by each of the vortices independently. Let the circulation around the first vortex when it is in isolation (say rotating counterclockwise and on the left when looking down on the 2-d flow field) be equal to Γ_1. The circulation around the second vortex when it is in isolation (rotating counterclockwise and on the right when looking down on the 2-d flow field) be equal to Γ_2.

   a) Compute the velocity of the center of the right vortex due to the velocity field induced by the left vortex. Use a coordinate system where $\hat{y}$ is up-the-page.

   b) Compute the velocity of the center of the left vortex due to the velocity field induced by the right vortex. Use a coordinate system where $\hat{y}$ is up-the-page.

   c) You should have found that your answers to (a) and (b) were in opposite directions. On the line segment connecting the vortex centers, there is a changing fluid velocity as a function of distance x from the left vortex. Find the value of x (in terms of Γ_1, Γ_2, and h) where the velocity field due to both vortices is zero. This point is actually a stationary point in the velocity field, and the two vortices "orbit" this point. As such, sometimes this point is called the "center of gravity" of the vortex system.

   d) Check your answer to part (c) to verify that when $\Gamma_1 = \Gamma_2$, you get the clear limiting case that the center of gravity would be $h/2$.

   e) Use a computer program to plot x/h as a function of Γ_1/Γ_2. Make sure the final figure is pretty professional looking. Make your x-axis logarithmic, and consider values of Γ_1/Γ_2 ranging from 10^{-5} to 10^5.

4. We're starting to get into the final portion of the semester. I know y'all have a large diversity of interests, and – though I have made efforts to try and give you a relatively comprehensive treatment that has introduced us to a wide variety of topic important within the study of fluids – there are topics that I'd love to cover that we won't have time for. Below, I list a variety of additional topics we may discuss this semester. I'd like you to rank these from 1 (least interesting to you) to 6 (most interesting to you). Although I can't promise I will necessarily follow the classes preferences, I will try to take them into account for the road we still have ahead. I don't think it is reasonable for us to get into all of these topics, because then we could only spend about one day per topic, which isn't enough to do any of these justice.

   a) Further discussion of topics related to vorticity (most notably lift and potential vorticity)
   b) Detailed study of low Reynolds-number flows (there's a really cool video we're going to watch either way, but we can spend a couple of lectures on this if there's general interest).
   c) Detailed study of waves (interfaces between two different fluids of different densities)
   d) Potential flows (lots of analogues to electricity and magnetism, but mostly useful for 2d systems).
   e) Turbulence (honestly, we could spend the entire rest of the semester on this one alone)
   f) Instabilities (kind of a vague title, but this includes a number of important flow structures that show up in a variety of contexts)

5. If there's any topic we have not yet talked about and ISN'T on the above list that you'd like to talk about in depth, please list it here.