Assignment XI, PHYS 301 (Classical Mechanics)
Spring 2015
Due 4/24/15 at start of class

1. (Should be no surprise): Three point masses represent a rigid body; the position/magnitude of the masses are as follows:

$$\begin{aligned} m_1 &= m \qquad \langle 0, 2a, a\rangle \\ m_2 &= m \qquad \langle 0, a, 2a\rangle \\ m_3 &= 2m \qquad \langle a, 0, 0\rangle \end{aligned}$$

   a) Write down the full Inertia tensor I for this system.
   b) Find the principle moments of inertia for this system.
   c) Find the orthogonal principle axes for this rotation and explicitly verify they are orthogonal. [How do you show that three axes are mutually orthogonal? Either: (i) show $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$ or (ii) show $\vec{v}_1 \times \vec{v}_2 = \lambda \vec{v}_3$ with λ some real constant.]

2. A uniform rectangular plate has corners at the origin, $a\hat{x}$, $2a\hat{y}$ and $a\hat{x} + 2a\hat{y}$. The plate has total mass M, so its mass density per unit area is $\sigma = \frac{M}{2a^2}$.

   a) Write down the full Inertia tensor I for this system (with respect to the origin). (Use Ms, not σs, in your answer).
   b) Find the principle moments of inertia with respect to the origin for this system. (Note – you do NOT have to find the principle axes)! (Again, use Ms, not σs, in your answer).

3. A thin (pseudo-1-dimensional) rod of length L lies between the origin and $L\hat{x}$. The linear mass density of this rod can be written as:

$$\lambda(x) = \lambda_\circ \left(1 + 2\frac{x^3}{L^3}\right)$$

   a) What is the mass of the rod?
   b) What is the moment of inertia of the rod with respect to the $\hat{y}$ axis going through the origin?
   c) What is the moment of inertia of the rod with respect to an axis parallel to the $\hat{y}$ axis that goes through the point $L\hat{x}$?
   d) Were is the center of mass of the rod?
   e) The rod is balanced on its lighter end on a rough floor and then tips over; the point in contact with the Earth doesn't move (it pivots about its end). What is the rod's angular velocity when it hits the floor? (You may neglect the rotation of the Earth).
   f) If the rod was balanced on its heavy end and the same thing happened as in part (e) above, would the angular velocity be larger or smaller when the rod hits the floor? Justify your answer either via solid reasoning or calculation.

4. An incompressible but moldable blob of matter of total mass M is to be situated between two flat horizontal planes at $z = 0$ and $z = 10$ so that the moment of inertia around the z axis is as small as possible. What shape should the blob take?

5. An ant crawls with constant speed in a circular path of radius R on a turntable rotating with constant angular speed Ω. The circular path is concentric with the center of the turntable. If the mass of the ant is m and the coefficient of static friction with the surface of the turntable is μ_s, how fast – relative to the turntable – can the ant crawl before it starts to slip if it travels:

   a) In the direction of rotation. (Solve this problem in an inertial reference frame. – This is, by most accounts, the easier choice.)

   b) In the direction opposite of rotation. (Solve this problem in the rotating reference frame. This is harder for most people.)

   Note – if you like, you can check your answers by solving each part in the other reference frame. You should get the same answer once you account for the relative motion between the frames.

6. A smooth (frictionless) rod of length ℓ rotates in a plane with constant angular velocity ω about an axis fixed at the end of the rod and perpendicular to the plane of rotation. A bead of mass m is initially positioned at the stationary end of the rod and given a slight push such that its initial speed directed down the rod is $v_\circ = \omega\ell$. Calculate how long it takes for the bead to reach the other end of the rod.

7. A cannonball is fired straight up with initial speed $v_\circ$.

   a) Assuming g is constant and neglecting air resistance, show that the cannonball will hit the ground displaced from the initial point of upward motion by an amount:

   $$\frac{4\Omega v_\circ^3 \cos\theta}{3g^2}$$

   where θ is the latitude and Ω is the Earth's angular speed.

   b) What direction will the displacement be? (North, South, East, or West?) (You may assume $\theta > 0$ / Northern Hemisphere).

   c) Let's see what this means in terms of a realistic system. Look up the latitude for Charleston. Let's say you launch a 12 inch cannonball straight up on the battery with velocity 1769 feet per second. (I know the units suck. Blame the gun makers). How far is this displaced from the launch point upon landing? (Would it destroy the cannon?) (Assume no air resistance).

8. In addition to the Coriolis and Centrifugal force, there is a third "fictitious" force associated with a rotating system – sometimes called the "azimuthal force" or "Euler force". This force has magnitude $|m\vec{r}\times\dot{\vec{\Omega}}|$. Find the ratio of this Azimuthal force to the Coriolis force on Earth. Assume you are traveling in a car moving 30 m/s to the North at a latitude of 45 degrees North. You may assume the Earth is spherical. To help you estimate $\dot{\vec{\Omega}}$, note that the axial tilt of the earth changes from about 22.1° to about 24.5° and back periodically. It takes about 40,000 years for a full cycle. For simplicity, assume that the change in angle occurs at a constant rate over the 20,000 years that it takes to change the 2.4 degrees. [In reality, it is more of a sinusoidal change, but let's just use the constant-rate assumption to get an order-of-magnitude estimate]. You may assume that the length of the day stays the same over time-scales of 20,000 years, and make any necessary assumptions you need to about Longitude so that the Azimuthal force is as large as possible.