Homework 11, PHYS 415 (Fluid Mechanics)
Spring 2019
Due Thursday 11 April 2019 at Beginning of Class

As always, turn in your legible and annotated work on separate paper.

1. In class, we have (or will) give the expression for the (phase) velocity of a gravity wave at the interface between a fluid and air (e.g. a water wave) in the absence of surface tension to be:

$$c_{\text{phase}} = \sqrt{\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi H}{\lambda}\right)}$$

where λ is the wavelength of the wave, g is the acceleration of gravity, and H is the depth of the unperturbed fluid layer. A more nuanced treatment that accounts for the influence of surface tension gives a revised expression:

$$c_{\text{phase}} = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}\right)\tanh\left(\frac{2\pi H}{\lambda}\right)}$$

a) Finding a closed-form expression for the wavelength where the minimum phase velocity occurs when surface tension effects are not negligible is messy; originally I was going to have you do it, but it is a bit too gnarly for me to ask for. Rather, I'll give you the answer – if you assume $H \gg \lambda$, you find the wavelength of minimum phase speed to be $\lambda_m = 2\pi\sqrt{\frac{\sigma}{\rho g}}$. Find an expression for $c_{\text{phase}}(\lambda_m)$ and find λ_m and the associated $c_{\text{phase}}(\lambda_m)$ for liquid water on earth. You may assume $H \gg \sqrt{\frac{\sigma}{\rho g}}$.

b) Show that the ratio of c_{phase} including surface tension to c_{phase} not including surface tension for water reduces to approximately:

$$1 + \frac{1.45 \times 10^{-4}\ \text{m}^2}{\lambda^2}$$

when $4\pi^2\sigma \ll \rho g\lambda^2$.

c) Given the above expression, find the wavelength where the error in ignoring surface tension results in a 3% error in the computed velocity of the wave.

d) Make a plot of c_{phase} (including the surface tension effects) as a function of λ for water of depth of 3000 meters (approximately the average depth of the ocean), 100 meters, and 1 meter. Make a publication quality figure, and make both axes logarithmic so we can clearly see behaviors as a function of λ ranging all the way from 10 microns to about 10 km. I'm going to be pretty picky here, so make sure it looks good.

e) Based on your plot, is it possible for the crest of a water wave to break the sound barrier on Earth?

2. (This problem is a modified version of a problem Dr. Fragile gave on his midterm when he taught this course). Consider a two-dimensional flow with velocity components:

$$v_x = U\cos(kx)\mathrm{e}^{-ky}$$
$$v_y = -U\sin(kx)\mathrm{e}^{-ky}$$

where U and k are constants.

   a) Is this an incompressible flow?
   b) Find $\oint \vec{v}\cdot \mathrm{d}\vec{\ell}$ around a circle of radius R centered around the origin and lying in the $x-y$ plane.
   c) Find the vorticity of this velocity field.

3. We have a final exam coming up in a couple weeks. One of the best thing you can do to study is to anticipate what sorts of things I might ask and make sure you can solve them. This problem is to engage in that process a little bit.

   a) Write a problem that would, in your opinion, be fair for me to ask on your final exam. This problem should not be trivially easy (nothing where you just ask the test-taker to just recall a numerical value or equation; we want to test understanding and problem-solving skill, not merely the ability to memorize material). *Note: I may end up using some of your suggested problems on the final! If I use a problem you suggest, you'll earn a little bit of extra-credit on your semester homework grade. As a bonus, since you solve your own problem in part (b) below, if I use your suggested problem presumably you also have at least one final exam question in the bag!*
   b) Solve the problem you propose in part (a) correctly.

   Note! I know a lot of you work collaboratively on homework. That's ok. However, I expect everyone's suggested problem above to be meaningfully different from everyone else's suggested problem. If there are duplicates, neither of you will earn credit for this homework problem. Note that the same essential problem with different fluids and/or different numerical parameters will be treated as the same problem for this evaluation.