1 "Perfect Randomness" – The Poisson Process

Independent of whether a statistically homogeneous or inhomogeneous approach is used to describe a statistical data-set, the gold-standard of perfect randomness remains the Poisson process.

By construction, a Poisson process is homogeneous. It is as random as randomness allows; there are no extraneous clumps or clusters, but there isn't an anomalous lack of clumps or clusters, either.

A Poisson process is predicated on three assumptions (expanded from the 1-dimensional construction given in the Cramér and Leadbetter (2004) paraphrase of Khintchine 1960):

- The probability that k events will occur in a volume of size V depends on k and the magnitude of V, but not on the location of V.
- The events occurring in disjoint volumes are mutually independent random variables.
- The probability that more than one event occurs in a small volume $\mathrm{d}V$ is $o(\mathrm{d}V)$ as $\mathrm{d}V \rightarrow 0$.

From this, we can directly derive the measurable properties of a Poisson process (e.g. its distribution function). Following Cramér and Leadbetter (2004) and proceeding in 1-dimension, we can write:

$$p_0(t_1 + t_2) = p_0(t_1)p_0(t_2) \tag{1}$$

for any time intervals t_1 and t_2, where $p_0(t)$ is the probability of finding no particles in an interval of duration t. This implies, then, that:

$$p_0(t) = \exp(-\lambda t) \tag{2}$$

where λ is positive.

We can use some further tricks from Cramér and Leadbetter (2004) to get the distribution function as well. For small dt this means that:

$$p_0(\mathrm{d}t) = 1 - \lambda \mathrm{d}t + o(\mathrm{d}t) \tag{3}$$

but if $p_{k>1}(\mathrm{d}t)$ is negligible as required above, we can then say that:

$$p_1(\mathrm{d}t) = 1 - p_0(\mathrm{d}t) + o(\mathrm{d}t) = \lambda \mathrm{d}t + o(\mathrm{d}t) \tag{4}$$

In general, then, for arbitrary t and small dt we have that:

$$p_k(t + \mathrm{d}t) = p_0(\mathrm{d}t)p_k(t) + p_1(\mathrm{d}t)p_{(k-1)}(t) \tag{5}$$
$$p_k(t + \mathrm{d}t) = (1 - \lambda \mathrm{d}t)p_k(t) + \lambda \mathrm{d}t p_{(k-1)}(t) + o(\mathrm{d}t) \tag{6}$$

subtracting $p_k(t)$ from both sides and dividing by dt we get an equation for the derivative of $p_k(t)$:

$$\frac{p_k(t + \mathrm{d}t) - p_k(t)}{\mathrm{d}t} = p'_k(t) = \lambda \left[p_{(k-1)}(t) - p_k(t)\right] \tag{7}$$

The solution of this differential equation is found to be:

$$p_k(t) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t) \tag{8}$$

Equations 2 and 8 are two of the three "routes to a Poisson process". The third one is a little more qualitative in nature.

Given a particle placed in $[t_1, t_2]$, what is the probability it is placed in $[t_1, t]$ (notated by $\mathcal{P}(t)$) with $t_1 \leq t \leq t_2$? We know (since it is explicitly placed in the interval), that $\mathcal{P}(0) = 0$, $\mathcal{P}(t_2 - t_1) = 1$, and that $\mathcal{P}$ is independent of t_1 and t_2, but is a function of t ...