Assignment V, PHYS 409 (Electromagnetism I)
Fall 2019 Due Tuesday, October 1st, 2019 at beginning of class

Please supply your complete, legible, and well organized solutions on separate paper. No Mathematica, calculators, or other technological aids should be used (or are necessary) to complete this homework assignment.

1. Let a charge $+q$ exist at the position $a\hat{x}+b\hat{y}$. In this system, both the positive x axis and the positive y axis are grounded. (a and b are positive constants).

   a) Find the potential everywhere within the region where $x>0$ and $y>0$. (e.g. find $V(x,y)$ with the understanding that x and y are both positive.

   b) Assume that $b \gg a$. Find V at the point $a\hat{x}+2b\hat{y}$ for (i) the case as described above (with conductors on the x and y axes), and (ii) the simpler situation where there is just a single point charge of $+q$ at the point $a\hat{x}+b\hat{y}$ (same as the problem), but there are no image charges or grounded conductors – just a single point charge. (For case 1, you will want to carry out an approximation. Remember that if $b \gg a$, you can treat a/b as a small parameter and expand in a Taylor series.)

   c) Is the potential larger for case (i) or (ii)? (You may assume $q>0$). Explain why this makes sense.

2. Let the entire $x-y$ plane be grounded (just like in the “classic” image problem). Let there be a charge of $+q$ at $3a\hat{z}$ and another charge of $-2q$ at $5a\hat{z}$. (a is an arbitrary positive constant, q may be assumed to be positive).

   a) Find the potential everywhere within the region $z>0$. (e.g. find $V(x,y,z)$ with the understanding that $z>0$).

   b) Find the total force on the charge $+q$ due to the other charge and the presence of the infinite conducting plane. (You may assume that we're just looking for the instantaneous force here, so nothing is actually allowed to move – since that changes V which changes E which changes the force, etc.) You should simplify your answer to a single term! (No calculators. I want to make sure you remember your arithmetic!)

   c) Find $\sigma(x,y)$ on the surface of the infinite conducting plane. (Note, even my answer is a bit messy, but I was able to get it down to two terms).

   d) Integrate the relationship you found for $\sigma(x,y)$ in part (c) to find the total charge induced on the surface of the infinite conducting plane.

(MORE ON BACK!)

3. Verify (via direct substitution and manipulation) that, for constant A and B, $R(r) = Ar^{\ell} + \frac{B}{r^{\ell+1}}$ is a valid solution to the differential equation:

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \ell(\ell+1)$$

4. Use the Rodrigues formula to find $P_4(x)$. Since the answer is right in your book, you can easily verify your solution. Therefore, you must show all of your work on this problem to garner full credit.

5. The claim is made that the Legendre Polynomials are Orthogonal, e.g. they follow the relationship specified by

$$\int_{-1}^{1} P_{\ell}(x)P_{\ell'}(x)\mathrm{d}x = \begin{cases} 0 & \ell' \neq \ell \\ \frac{2}{2\ell+1} & \ell' = \ell \end{cases}$$

Let's verify this relationship for a few pairs of ℓ and ℓ'. (For this problem, we're just plugging in a couple different combinations of P_{ℓ} and $P_{\ell'}$ and verifying that we get the expected value from the integral. You may use the table of $P_{\ell}(x)$ in your book; no need to use the Rodrigues formula to generate the Legendre polynomials for THIS problem).

a) Verify (through direct substitution and carrying out the integration) that you get the expected value when you integrate $\int_{-1}^{1} P_2(x)P_3(x)\mathrm{d}x$.

b) Verify (through direct substitution and carrying out the integration) that you get the expected value when you integrate $\int_{-1}^{1} P_2(x)P_2(x)\mathrm{d}x$.

c) Verify (through direct substitution and carrying out the integration) that you get the expected value when you integrate $\int_{-1}^{1} P_3(x)P_3(x)\mathrm{d}x$.

d) Verify (through direct substitution and carrying out the integration) that you get the expected value when you integrate $\int_{-1}^{1} P_0(x)P_2(x)\mathrm{d}x$.

e) Verify (through direct substitution and carrying out the integration) that you get the expected value when you integrate $\int_{-1}^{1} P_1(x)P_3(x)\mathrm{d}x$.