Assignment X, PHYS 409 (Electromagnetism I)
Fall 2019 Due Thursday, November 14, 2019

1. I have to give you at least one Biot-Savart problem. I could give you a problem that can't be solved via other methods like Ampere's law or through multipole expansion – but those tend to be rather ugly. Alternatively, I could give you a problem with enough symmetry to solve via other methods but mandate your solution method. In this case, I will choose the latter option. You will be able to find solutions to this problem if you google, but most of them won't be using the method I mandate here. The online solutions may help you check your answer, but I specifically want you to do the problem in the way I outline below.

   Consider a disk of radius R carrying total charge Q evenly distributed across its surface. The disk rotates with uniform angular velocity ω through its center in such a way that the angular velocity vector is in the $\hat{z}$ direction. The charges on the surface stay tied to their locations relative to the disk, thus this rotation induces a current in the external non-rotating frame.

   a) We'll walk you through this sequentially, so some of the steps will be pretty straightforward. What is the surface charge density σ on the disk as a function of Q and R?

   b) What is the surface current density $\vec{K}$? (Remember that $\vec{K}$ is a vector and work in cylindrical coordinates).

   c) To calculate the magnetic field at some other location, we need to use the Biot-Savart law for surface currents

   $$\vec{B}(\vec{r}) = \frac{\mu_\circ}{4\pi} \oint\!\!\oint \frac{\vec{K} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \mathrm{d}a'$$

   (In case you haven't seen it before, $\mathscr{r}$ is script-r; that's the best I can do with LaTeX). What is the magnitude of $|\vec{K} \times \hat{\mathscr{r}}|$ in terms of Q, ω, s, R, and/or z for a point on the z axis?

   d) In general, $\vec{K} \times \hat{\mathscr{r}}$ has components in both the $\hat{z}$ and $\hat{s}$ direction. However, from symmetry, we know that only the $\hat{z}$ component of $\vec{K} \times \hat{\mathscr{r}}$ will need to be integrated since the other component will cancel from a symmetric point on the other side of the disk. What is the z component of $\vec{K} \times \hat{\mathscr{r}}$ as a function of Q, ω s, R, and/or z for a point on the z axis?

   e) Given your methods so far, write down the appropriate double integral (with proper limits) that needs to be evaluated to find $\vec{B}(z)$ in terms of Q, ω, s, R, and/or z.

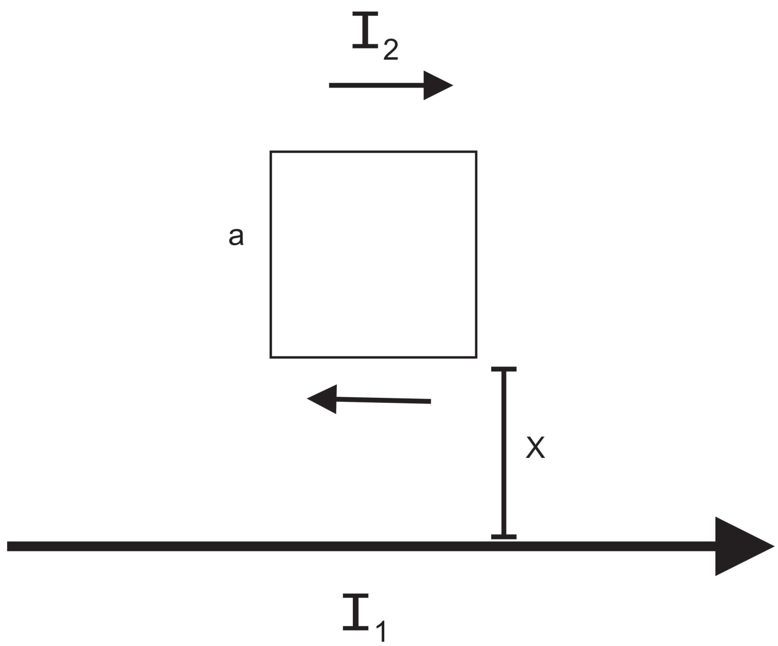
   f) Evaluate the double integral developed in part (e) above to find $\vec{B}(z)$. You are not allowed to look anything up or use technology, but I will give you a little help:

   $$\int \frac{x^3 \mathrm{d}x}{(x^2+z^2)^{(n+1)}} = \frac{-1}{2(n-1)(x^2+z^2)^{n-1}} + \frac{z^2}{2n(x^2+z^2)^n}$$

   for any real n. You may also assume $z \geq 0$. Simplify your answer.

   g) Evaluate your answer to part (f) above for $z = 0$ to find the magnetic field in the middle of the disk. (It won't be zero).

   h) Find an approximate expression for $\vec{B}(z)$ as $z \to \infty$? (You can't just write 0 here!) Keep all terms involving powers of z less than or equal to inverse cubed.



2. Above, view a graphical interpretation of this problem. A wire carries constant current I_1 to the right. Another wire square with side-length a carries clockwise current I_2.

   a) Find the force on the square due to the magnetic field created by the wire in terms of I_1, I_2, x, a, and/or fundamental constants.

   b) A second wire, placed parallel to the wire carrying current I_1, carries the same magnitude of current as the square I_2 (but carries it from right to left). For this part of the problem, let $x = 2a$. If the net force on the square is zero due to the forces resulting from both currents, where must this second wire be placed?

   c) Confirm your answer to part (b) makes sense in the case where $I_1 = I_2$.

3. Let there be an infinitely long straight cylindrical wire of radius a that carries a volume current density $\vec{J}(s) = J_\circ e^{-s^2/a^2}\,\hat{z}$ for some constant $J_\circ$.

   a) Find the Magnetic field in the region $s < a$. (Note! You should not need to use Mathematica or any other computer resource to carry out these integrals!)

   b) Find the Magnetic field in the region $s > a$. (Note! You should not need to use Mathematica or other computer resources to carry out these integrals!)

   c) Use your answer to part (a) to confirm that $\vec{\nabla} \times \vec{B} = \mu_\circ \vec{J}$ inside the wire.

   d) Use your answer to part (b) to confirm that $\vec{\nabla} \times \vec{B} = 0$ outside the wire (where $\vec{J}$ is zero).

   e) Use your answer to part (a) to confirm that $\vec{\nabla} \cdot \vec{B} = 0$ inside the wire.

   f) Use your answer to part (b) to confirm that $\vec{\nabla} \cdot \vec{B} = 0$ outside the wire.

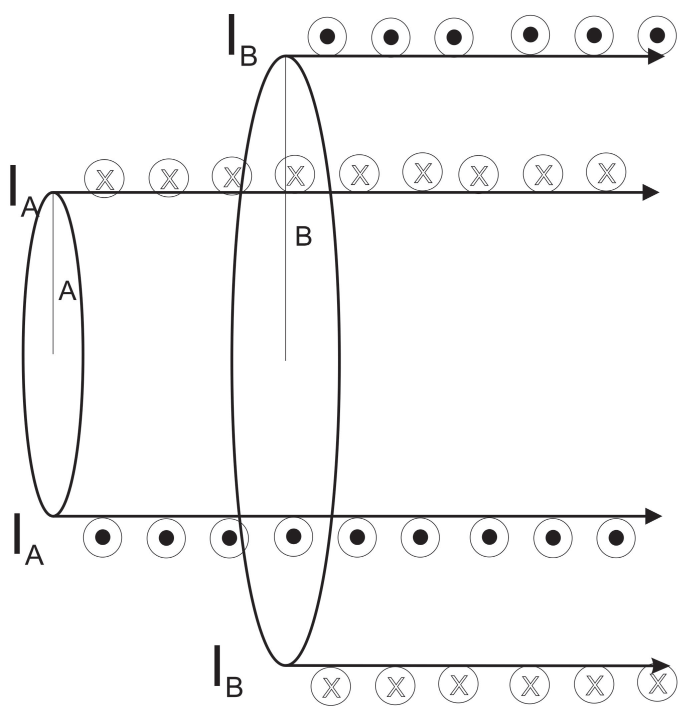
4. An infinite slab runs from $z = -a$ to $z = a$ and carries a volume current density $\vec{J} = kz^2\,\hat{y}$.

   a) What must the units of k be so that $\vec{J}$ is a valid volume current density?

   b) Find the magnetic field in the region $0 < z < a$.

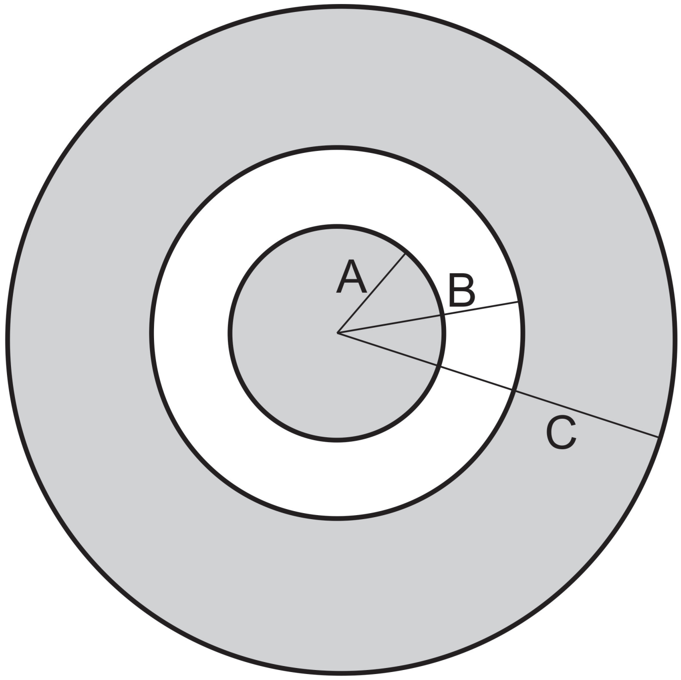
   c) Find the magnetic field in the region $z > a$.

   d) Let's say a uniform surface current runs on the edges of the slab (at $z = a$ and $z = -a$) that exactly cancels the magnetic field in the range $z > a$. What is $\vec{K}$?



5. Let there be two infinite coaxial concentric solenoids with radii A and B carrying currents I_A and I_B respectively (as shown above, the currents run in opposite directions. Imagine the wires wrapping around the cylinders to generate the currents shown going into and coming out of the paper). Let the inner solenoid have n_a turns per unit length and let the outer solenoid have n_b turns per unit length. Find the magnetic field everywhere ($s < A$, $A < s < B$, and $s > B$).

MORE ON BACK!



6. An infinite coaxial cable has a center region (where $s \le A$) that carries a current I_{inner} into the page. The outer region (where $B \le s \le C$) carries a (potentially different) current I_{outer} out of the page. Unlike what we would normally expect for conductors, however, somehow the current is distributed *uniformly throughout the volumes of each current-carrying region.* (Maybe this coaxial cable is made of a dielectric for some strange reason). You may assume that there is a vacuum between the current-carrying regions and outside the cable.
   a) $\vec{J}$ inside the center region is rather straightforward to find; it is $\vec{J}(s) = \frac{I_{\text{inner}}}{\pi A^2}\hat{z}$. What is $\vec{J}$ for $B \le s \le C$? (This isn't meant to be too hard).
   b) Find the magnetic field everywhere (inside the inner region, between the currents, inside the outer current-carrying region, and outside the cable).
   c) Use your answer to part (a) to confirm that $\vec{\nabla} \times \vec{B} = \mu_\circ \vec{J}$ inside both current-carrying regions.

7. If you calculate the magnetic dipole moment of the system described in problem number (1) of this assignment, you should find $\vec{m} = \frac{\pi\omega\sigma R^4}{4}\hat{z}$.
   a) Given that $\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_\circ \vec{m}\times\hat{r}}{4\pi r^2}$, find $\vec{A}_{\text{dip}}(\hat{r})$ for this system by explicitly taking the requisite cross product. Since there is no easy/general formula for cross product in spherical, you will want to write $\hat{z}\times\hat{r}$ in Cartesian and then convert the Cartesian result to Spherical. (There is a shorter way to do this by appealing to an equation a couple pages later in your text, but I want you to actually compute the cross product here).
   b) Take the curl of the result you found above to obtain the approximate magnetic field for this system. Leave your answer in terms of σ, ω, R, and fundamental constants and in spherical coordinates.
   c) Use your answer to 1(a) and evaluate your answer to the above at $\theta = 0$ to verify this matches your answer to problem 1(h)