Assignment V, PHYS 459 (Cloud and Precipitation Physics)
Fall 2019 Due Thursday, September 26th, 2019 at Beginning of Class

1. In class, we went into some detail to calculate the terminal velocity of an aerosol particle falling in still air. Eventually, we developed the equation:

$$|\vec{F}_{\text{drag}}| = \frac{\rho_f \pi D_p^2 C_D v_t^2}{8C_C}$$

with:

$$C_C = 1 + \mathrm{Kn}\left(1.257 + 0.4e^{-1.1/\mathrm{Kn}}\right)$$

$$C_D = \begin{cases} \frac{24}{\mathrm{Re}} & \mathrm{Re} < 0.1 \\ \frac{24}{\mathrm{Re}}\left(1 + \frac{3}{16}\mathrm{Re} + \frac{9}{160}\mathrm{Re}^2 \ln(2\mathrm{Re})\right) & 0.1 \le \mathrm{Re} \le 2 \\ \frac{24}{\mathrm{Re}}\left(1 + 0.15\mathrm{Re}^{0.687}\right) & 2 \le \mathrm{Re} \le 500 \\ 0.44 & 500 \le \mathrm{Re} \le 500000 \end{cases}$$

Show that this general case reduces to the Stokes' drag equation in the limit $\mathrm{Kn} \to 0$ and $\mathrm{Re} \to 0$. (Note that saying the drag is approximately 0 in that case because the Reynolds number is small isn't good enough here. You should show after plugging in C_C and C_D that the expression you get looks exactly like the Stokes' Drag equation $F_{\text{drag}} = 3\pi\mu D_p v_t$.) You will need to use $L = D_p$ as the characteristic length of the flow.

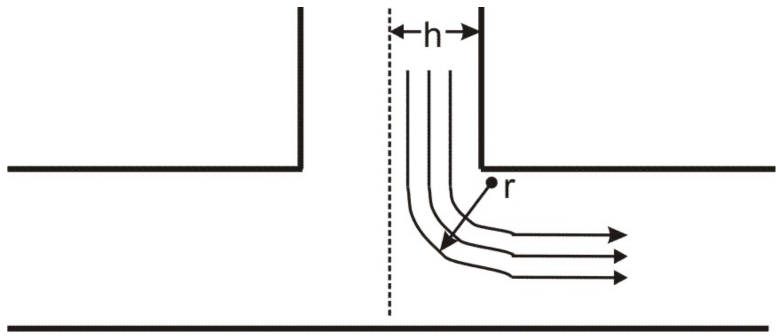
2. In class, I asserted that the equation:

$$\tau \frac{\mathrm{d}v_z}{\mathrm{d}t} = \tau g - v_z$$

With initial condition $v_z(t=0) = 0$ can be solved via:

$$v_z(t) = \tau g\left(1 - e^{-t/\tau}\right)$$

   a) Verify that $v_z(t=0) = 0$ in the above formula.
   b) Verify this proposed solution satisfies the differential equation.
   c) Find $v_z(t \to \infty)$
   d) How long (in terms of τ) would it take for an aerosol starting at rest to reach a fall-speed of $\alpha v_z(t \to \infty)$ (i.e. how long would it take to be $\alpha \times 100$ percent of the way to the terminal velocity?)
   e) Given your answer to part (d), how long would it take a 100 nm diameter aerosol particle (with density 1300 kg/m^3) to reach 95% of its terminal fallspeed? (Use 1.2 as the Knudsen number for a 100 nm aerosol particle).
   f) How far would the aerosol particle have fallen in part (e) in the time it takes to reach 95% of its terminal fallspeed?



3. For the simplified rectangular impactor nozzle drawn above, you may assume that the flow is uniform near the exit and that the streamlines of airflow are arcs of a circle with centers at the marked point. Some particles in the flow will move across streamlines and may be deposited on the surface.

   a) Show that particles of a given diameter, D_p, will move with constant radial velocity $v_r = (\tau U^2)/r$ where r is the radius of curvature of the streamline, τ is the relaxation time, and U is the speed of the gas. (You may assume U is constant).

   b) Show that in traveling along the arc, the particle is displaced by a total radial distance $\Delta = (\pi/2)\tau V$.

   c) Show that the fraction of particles that are deposited is $f = \frac{\pi\tau U}{2h} = \frac{\pi}{2} St$ where St is the Stokes' number.

4. To calibrate a rain-sensing device that we have (a 2-dimensional video disdrometer), we have a process by which we throw spherical steel ball-bearings through the device from a known height. I'm not 100% sure the exact height, but – for the sake of this problem – let's say that it is 60 cm above the sensing surface. Let's say we are dropping 10 mm spheres made out of solid steel (density of 8050 kg/m^3), initially at rest (hence "dropping") from a height of 60 cm through air (near the surface of the Earth). For these spheres, we ignore both wind and drag. Let's make sure this is reasonable.

   a) Assuming $C_D = 0.44$, what is the terminal velocity (in air) of these spheres?

   b) If you neglect air resistance entirely, the downward velocity of a falling object is $v = v_\circ + gt$ and, in particular, if dropped, you have $v = gt$. Use the formula derived in class to find the numerical ratio $v(t)/(gt)$ for these spheres for (i) $t = 0.1$ second, (ii) $t = 0.5$ seconds, (iii) $t = 1.0$ seconds, (iv) $t = 2.0$ seconds, (v) $t = 5.0$ seconds, and (vi) $t = 10$ seconds. (I personally used MATLAB to help me out here. Saved me a bunch of computation. You may solve this however you'd like.)

   c) Based on your answers to (a) and (b) (or other information, if necessary), explain why we don't have to factor in drag for these spheres.

5. Use your favorite computer algebra system/coding language/computational resource to draw a log-log plot of F_{drag} as a function of particle size. Have curves for Stokes' drag (dotted line) and corrected Stokes' drag (solid line) as presented in class. You may assume that the fluid is air (near the surface of the Earth) at 10°C, the fall velocity is 10 m/s, and have the x-axis (particle size) range from 0.1 nm to 1 cm. You may assume no slip-correction is necessary (e.g. $C_C = 1$). Note that this is 8 orders of magnitude difference in size, so you don't want to have a step size of 0.1 nm unless you have a week of computer time to kill. (Ask in class for hints!) In addition to the graph, please turn in your code/mathematica session/maple desktop/matlab code/excel spreadsheet/etc.

6. Assume the temperature at ground level is 300K. Also assume that we're talking about diffusion in air.

    a) Calculate the diffusion constant D for a 1 μm aerosol particle. (You may assume $C_C = 1$ for this aerosol).
    b) How long would it take a collection of 1 μm aerosol particles to diffuse from the center to the edges of a basketball in still air at 300 K? (In other words, how long until the RMS distance traveled by each aerosol would match the radius of a basketball?)
    c) What would C_C be for a 50 nm diameter aerosol? Use $C_C = 1+\mathrm{Kn}(1.257+0.4\mathrm{e}^{-1.1/\mathrm{Kn}})$ with $\ell = 8\times10^{-7}$ m.
    d) What is the diffusion constant D for the 50 nm aerosol particle in part (c)?
    e) How long would it take a collection of 50 nm aerosol particles to diffuse from the center to the edges of a basketball in still air at 300 K? (In other words, how long until the RMS distance traveled by each aerosol would match the radius of a basketball?)

7. We're going to be talking about diffusion of an ensemble of particles, which will enable us to do some fancier statistical mechanics. I'm going to have you push through one of the calculations underpinning that here. Consider the function $n(x,t) = n_\circ + \frac{\Delta N}{(4\pi D t)^{1/2}} \exp[-x^2/(4Dt)]$.

    a) We will verify that $n(x,t)$ above is a valid solution to the differential equation $\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}$. First, calculate $\frac{\partial n}{\partial t}$
    b) Now calculate $\frac{\partial n}{\partial x}$
    c) Now calculate $\frac{\partial^2 n}{\partial x^2}$
    d) Now multiply $\frac{\partial^2 n}{\partial x^2}$ by D and show you get something equivalent to $\frac{\partial n}{\partial t}$.
    e) Consider the integral $\int_{-\infty}^{\infty}[n(x,t) - n_\circ]\mathrm{d}x$. Evaluate it. (You may have to look up the "error function").
    f) What is the value of the integral $\int_{-X}^{X}[n(x,t) - n_\circ]\mathrm{d}x$? (You may leave your answer in terms of error functions and/or complementary error functions.)

MORE ON BACK!!

8. I gave you (in class) the following expression for the Diffusion Constant of an aerosol particle: $D = \frac{kTC_C}{3\pi\mu D_p}$. (Note that this expression has units of m^2/s). Another route to the diffusion equation developed from kinetic theory gives:

$$D^* = \frac{\lambda^2}{2t_{\text{ave}}}$$

where λ is the mean free path between collisions for a particle, and t_{ave} is the mean time between collisions. In practice, both of these expressions are legitimate ways of writing the diffusion constant. The first method makes a bit more sense for aerosols (where there is a particle and a fluid medium that are clearly different from each other). Using D^* makes more sense if you're just talking about diffusion of a single gas into, say, a vacuum or something. This problem is about playing with the two different definitions of D simultaneously.

   a) For methane, $D = 1.78 \times 10^{-5}$ m^2/s. Assume $\lambda = 90$ nm. Find the mean time between intermolecular collisions for methane.

   b) What is the mean *instantaneous* speed for a methane molecule? (Hint....speed = distance/time).

   c) What is the root mean squared displacement from the origin for a methane molecule released at $t = 0$ at the origin and left to travel for 3 minutes?

   d) What is the actual distance that the methane molecule moved in this time? (Not the displacement, but the distance).

   e) Derive an expression for (root mean squared displacement from the origin)/(distance traveled) as a function of t. Leave your answer in terms of D, t, and v_{ave} (the average instantaneous velocity of a methane molecule).

   f) It is physically impossible for the diffusion distance to be larger than the distance traveled. This means that the expression you calculated in part (e) only can hold when $t > t_\circ$. Find the smallest possible value of $t_\circ$ for methane.

9. Under certain circumstances, the radiation inside the sun can be treated like a gas of photons with the mean free path between interactions nominally 1 mm. (We say interactions instead of collisions because we're really talking about absorption and re-emission here. Instead of "colliding" with other photons, a photon moves for nominally 1 mm before being absorbed by something. Then it is re-emitted. You may assume this absorption/re-emission process is instantaneous). (This is a rather crude model, but it works for some situations).

   a) Under this basic model, how long (on average) would it take a photon created at the center of the sun to diffuse out to the surface of the sun? (It is not, of course, the "same" photon – but let's say we're tracing the energy path instead of the photon's path. We're abusing language a bit, but the language is at least evocative). Leave your answer in years.

   b) What is the ratio:
   (root mean squared displacement from the center of the sun)/(distance traveled)
   just as the photon gets to the sun's surface?