Writing Velocity and Acceleration in Polar Coordinates via use of Complex Numbers

In class, we have explicitly shown how you can write expressions for velocity and acceleration in polar coordinates. Although I gave a purely geometric/trigonometric development in class, a more succinct (and, in my opinion, more elegant) route is via the use of complex numbers. This gets us to the same place as we got in class via a different method.

Complex Plane

If you've had a complex variables course or Methods of Applied Physics, you know that one common way to write complex numbers is to write them in the form $z = Re^{i\phi}$ where R is the magnitude of the complex number (e.g. $R = (z^*z)^{1/2}$) and ϕ is the so-called "phase-angle". Using Euler's identity $e^{i\phi} \equiv \cos\phi + i\sin\phi$, we can note that:

$$z = Re^{i\phi} = R\cos\phi + iR\sin\phi$$

If we plot the real part of z along the x-axis and the imaginary part of z along the y-axis, we have plotted a "number" in a plane. Note the table below:

	"Cartesian Perspective"	"Polar Perspective"
$\vec{r}$	$x\hat{x} + y\hat{y}$	$\lvert\vec{r}\rvert\hat{r}$ at $\phi = \tan^{-1}(y/x)$
z	$R\cos\phi + iR\sin\phi$	$Re^{i\phi}$

Thus, instead of doing funky manipulations on $\vec{r}$ and using geometry/trigonometry, here we use the parallelism between complex numbers and 2d representations of points to try to get to the expressions for $\dot{\vec{r}}$ and $\ddot{\vec{r}}$.

The convention we will use is that, when we write quantities without complex exponentials, we will interpret any pure real quantity as being in the $\hat{x}$ direction and any purely imaginary quantity as being in the $\hat{y}$ direction. Since $e^{i\phi}$ is our analog to $r\hat{r}$, anything with a complex exponential that is in the form of a purely real number times $e^{i\phi}$ will be interpreted to be in the $\hat{r}$ direction and anything that is in the form of a purely imaginary number times $e^{i\phi}$ will be interpreted to be in the $\hat{\phi}$ direction. (Note that since $i = \mathrm{e}^{i\pi/2}$, we can write $i\mathrm{e}^{i\phi} = \mathrm{e}^{i\pi/2}\mathrm{e}^{i\phi} = \mathrm{e}^{i(\phi+\pi/2)}$ which would be an advancement of $\pi/2$ beyond the direction of $\hat{r}$, which makes sense for the direction of $\hat{\phi}$).

Thus, our complicated trigonometry and geometry will become a matter of just using the chain rule and relabeling at the end. (It doesn't seem simpler once you add all this text, but once you get it, it is rather simple to do this).

Velocity in Polar

So we start by writing that our "position" will be represented by $z = r\mathrm{e}^{i\phi}$. We seek to find:

$$\begin{aligned}\dot{z} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(r\mathrm{e}^{i\phi}\right)\\ \dot{z} &= \dot{r}\mathrm{e}^{i\phi} + r\dot{\phi}\mathrm{e}^{i\phi}\\ \dot{z} &= \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}\end{aligned}$$

(Once you have the analogue set up, it is so fast!).

Acceleration in Polar

Doing the same technique, we have:

$$\begin{aligned}\ddot{z} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\mathrm{e}^{i\phi} + ir\dot{\phi}\mathrm{e}^{i\phi}\right)\\ \ddot{z} &= \left[\ddot{r}\mathrm{e}^{i\phi} + i\dot{r}\dot{\phi}\mathrm{e}^{i\phi} + i^2r\dot{\phi}^2\mathrm{e}^{i\phi} + ir\ddot{\phi}\mathrm{e}^{i\phi} + i\dot{r}\dot{\phi}\mathrm{e}^{i\phi}\right]\\ \ddot{z} &= \left(\ddot{r} - r\dot{\phi}^2\right)\mathrm{e}^{i\phi} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)i\mathrm{e}^{i\phi}\\ \ddot{z} &= \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\hat{\phi}\end{aligned}$$

Where the only really tricky part is making sure you properly apply the product and chain rules when taking the necessary derivatives.